

Central limit theorem for an additive functional of the fractional Brownian motion

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Abstract

We prove a central limit theorem for an additive functional of the d -dimensional fractional Brownian motion with Hurst index $H \in (\frac{1}{1+d}, \frac{1}{d})$, using the method of moments, extending the result by Papanicolaou, Stroock and Varadhan in the case of the standard Brownian motion.

Keywords: fractional Brownian motion, central limit theorem, local time, method of moments.

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1 Introduction

Let $\{B(t) = (B^1(t), \dots, B^d(t)), t \geq 0\}$ be a d -dimensional fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$. If $Hd < 1$, then the local time of B exists (see, for instance, [4, 5, 6]) and can be defined as

$$L_t(x) = \int_0^t \delta(B(s) - x) ds, \quad t \geq 0, \quad x \in \mathbb{R}^d,$$

where δ is the Dirac delta function. The above local time is jointly continuous with respect to t and x (see [4]). For any integrable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ one can easily show the following convergence in law in the space $C([0, \infty))$, as n tends to infinity

$$\left(n^{Hd-1} \int_0^{nt} f(B(s)) ds, t \geq 0 \right) \xrightarrow{\mathcal{L}} \left(L_t(0) \int_{\mathbb{R}^d} f(x) dx, t \geq 0 \right). \quad (1.1)$$

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In fact, making the change of variable $s = nu$, and using the scaling property of the fBm we see that the process $\left(n^{Hd-1} \int_0^{nt} f(B(s))ds, t \geq 0\right)$ has the same law as

$$n^{Hd} \int_0^t f(n^H B(u)) du = n^{Hd} \int_{\mathbb{R}^d} f(n^H x) L_t(x) dx = \int_{\mathbb{R}^d} f(x) L_t(n^{-H} x) dx, \quad t \geq 0.$$

From here it is straightforward to verify (1.1).

If we assume that $\int_{\mathbb{R}^d} f(x)dx = 0$, then we see $n^{Hd-1} \int_0^{nt} f(B(s))ds$ converges to 0. It is interesting to know if there is a $\beta > Hd - 1$ such that $n^\beta \int_0^{nt} f(B(s))ds$ converges to a nonzero process. This will be proved to be true. In order to formulate this result we introduce the following space of functions. Fix a number $\beta > 0$ and denote

$$H_0^\beta = \left\{ f \in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} |f(x)| |x|^\beta dx < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} f(x) dx = 0 \right\}.$$

For any $f \in H_0^\beta$, by Lemma 4.1, the quantity

$$\|f\|_\beta^2 := - \int_{\mathbb{R}^{2d}} f(x) f(y) |x - y|^\beta dx dy \quad (1.2)$$

is finite and nonnegative. The next theorem is the main result of this paper.

Theorem 1.1 *Suppose $\frac{1}{d+1} < H < \frac{1}{d}$ and $f \in H_0^{\frac{1}{H}-d}$. Then*

$$\left(n^{\frac{Hd-1}{2}} \int_0^{nt} f(B(s)) ds, t \geq 0 \right) \xrightarrow{\mathcal{L}} \left(\sqrt{C_{H,d}} \|f\|_{\frac{1}{H}-d} W(L_t(0)), t \geq 0 \right)$$

in the space $C([0, \infty))$, as n tends to infinity, where “ $\xrightarrow{\mathcal{L}}$ ” denotes the convergence in law, W is a real-valued standard Brownian motion independent of B and

$$C_{H,d} = \frac{2}{(2\pi)^{\frac{d}{2}}} \int_0^\infty w^{-Hd} (1 - \exp(-\frac{1}{2w^{2H}})) dw = \frac{2^{1-\frac{1}{2H}}}{(1-Hd)\pi^{\frac{d}{2}}} \Gamma\left(\frac{Hd+2H-1}{2H}\right).$$

Notice that $\frac{Hd-1}{2} > Hd - 1$ since $H < \frac{1}{d}$. When $d = 1$ and $H = \frac{1}{2}$, the above theorem is obtained by Papanicolaou, Stroock and Varadhan in [11] with $C_{\frac{1}{2},1} = 2$. On the other hand, the constant $C_{H,d}$ is finite for any $H > \frac{1}{d+2}$. We conjecture that our result also holds for $\frac{1}{d+2} < H < \frac{1}{d}$. But we have not been able to show our result in the case $H \leq \frac{1}{d+1}$. The main reason is that in the proof of Proposition 3.4 we need $H > \frac{1}{d+1}$ (see the Remark at the end of Section 3).

In the critical case $Hd = 1$ the local time does not exist. For the Brownian motion case ($H = \frac{1}{2}$ and $d = 2$), Kallianpur and Robbins [7] proved that for any bounded and integrable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$\frac{1}{\log n} \int_0^n f(B_s) ds \xrightarrow{\mathcal{L}} \frac{Z}{2\pi} \int_{\mathbb{R}^2} f(x) dx,$$

as n tends to infinity, where Z is a random variable with exponential distribution of parameter 1. A functional version of this result was given by Kasahara and Kotani in [9], where

they also proved the second order results when $\int_{\mathbb{R}^2} f(x) dx = 0$. The Kallianpur-Robbins law was extended to the fBm by Kôno in [10], and the corresponding functional version was obtained by Kasahara and Kosugi in [8]. However, second order results for the fBm in the critical case $Hd = 1$ have not been yet proved. On the other hand, we refer to Biane [3] for some extensions of these results to the case of functionals of k independent Brownian motions.

After some preliminaries in Section 2, Section 3 is devoted to the proof of Theorem 1.1, based on the method of moments. Throughout this paper, if not mentioned otherwise, the letter c , with or without a subscript, denotes a generic positive finite constant whose exact value is independent of n and may change from line to line. We use ι to denote $\sqrt{-1}$.

2 Preliminaries

Let $\{B(t) = (B^1(t), \dots, B^d(t)), t \geq 0\}$ be a d -dimensional fractional Brownian motion with Hurst index $H \in (0, 1)$, defined on some probability space (Ω, \mathcal{F}, P) . That is, the components of B are independent centered Gaussian processes with covariance

$$\mathbb{E}(B^i(t)B^i(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

The next lemma gives a formula for the moments of the increments of the process $\{W(L_t(0)) : t \geq 0\}$ on disjoint intervals, where W is a real-valued standard Brownian motion independent of B .

Lemma 2.1 *Fix a finite number of disjoint intervals $(a_i, b_i]$ in $[0, \infty)$, where $i = 1, \dots, N$ and $b_i \leq a_{i+1}$. Consider a multi-index $\mathbf{m} = (m_1, \dots, m_N)$, where $m_i \geq 1$ and $1 \leq i \leq N$. Then*

$$\begin{aligned} & \mathbb{E} \left(\prod_{i=1}^N [W(L_{b_i}(0)) - W(L_{a_i}(0))]^{m_i} \right) \\ &= \begin{cases} \left(\prod_{i=1}^N \frac{m_i!}{2^{\frac{m_i}{2}} (2\pi)^{\frac{m_i d}{4}} (m_i/2)!} \right) \int_{\prod_{i=1}^N [a_i, b_i]^{\frac{m_i}{2}}} \det(A(w))^{-1/2} dw & \text{if all } m_i \text{ are even} \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (2.1)$$

where $A(w)$ is the covariance matrix of the Gaussian random vector

$$\left(B(w_k^i) : 1 \leq i \leq N \text{ and } 1 \leq k \leq \frac{m_i}{2} \right).$$

Proof. It is easy to see that when one of m_i is odd, then the expectation is 0. Suppose now that all m_i are even. Denote by \mathcal{F}^B the σ -algebra generated by the fractional Brownian

motion B . Since W is a standard Brownian motion independent of B , we have

$$\begin{aligned}
& \mathbb{E} \prod_{i=1}^N [W(L_{b_i}(0)) - W(L_{a_i}(0))]^{m_i} \\
&= \mathbb{E} \left\{ \mathbb{E} \left(\prod_{i=1}^N [W(L_{b_i}(0)) - W(L_{a_i}(0))]^{m_i} \middle| \mathcal{F}^B \right) \right\} \\
&= \left[\prod_{i=1}^N \frac{m_i!}{2^{\frac{m_i}{2}} (m_i/2)!} \right] \mathbb{E} \prod_{i=1}^N [L_{b_i}(0) - L_{a_i}(0)]^{m_i/2} \\
&= \left(\prod_{i=1}^N \frac{m_i!}{2^{\frac{m_i}{2}} (2\pi)^{\frac{m_i d}{4}} (m_i/2)!} \right) \int_{\prod_{i=1}^N [a_i, b_i]^{m_i/2}} \det(A(w))^{-1/2} dw.
\end{aligned}$$

This completes the proof. \blacksquare

We shall use the following local nondeterminism property of the fractional Brownian motion (see [1]): for any $0 = s_0 < s_1 \leq \dots \leq s_n < \infty$ and $u_1, \dots, u_n \in \mathbb{R}^d$, there exists a positive constant k_H such that

$$\text{Var} \left(\sum_{i=1}^n u_i \cdot (B(s_i) - B(s_{i-1})) \right) \geq k_H \sum_{i=1}^n |u_i|^2 (s_i - s_{i-1})^{2H}. \quad (2.2)$$

This can also be written as

$$\text{Var} \left(\sum_{i=1}^n u_i \cdot B(s_i) \right) \geq k_H \sum_{i=1}^n \left| \sum_{j=i}^n u_j \right|^2 (s_i - s_{i-1})^{2H}. \quad (2.3)$$

We claim that the law of the random vector $(W(L_{b_i}(0)) - W(L_{a_i}(0)) : 1 \leq i \leq N)$ is determined by the moments computed in Lemma 2.1. This is a consequence of the following estimates. Fix an even integer $n = 2k$, and set $D_k = \{s \in [0, t]^k : 0 < s_1 < s_2 < \dots < s_k < t\}$. Let $A_k(s)$ be the covariance matrix of Gaussian random vector $(B(s_1), B(s_2), \dots, B(s_k))$. Then the local nondeterminism property (2.2) implies that

$$(\det A_k(s))^{-\frac{1}{2}} \leq c^k \prod_{i=1}^k (s_i - s_{i-1})^{-Hd} \quad \text{for some constant } c. \quad (2.4)$$

As a consequence of (2.1) and (2.4),

$$\begin{aligned}
\mathbb{E} [W(L_t(0))]^n &\leq c^k n! \int_{D_k} s_1^{-Hd} (s_2 - s_1)^{-Hd} \dots (s_k - s_{k-1})^{-Hd} ds \\
&= c^k n! \int_{\{0 < u_1 + \dots + u_k < t\}} \prod_{i=1}^k u_i^{-Hd} du \\
&= c^k n! t^{k(1-Hd)} \frac{\Gamma^k(1-Hd)}{\Gamma(k(1-Hd) + 1)}.
\end{aligned}$$

Therefore, $\mathbb{E} [W(L_t(0))]^n$ is bounded by $c^k n! / \Gamma(k(1-Hd) + 1)$, and this easily implies the desired characterization of the law of the increments of the process $\{W(L_t(0)) : t \geq 0\}$ on disjoint intervals by its moments.

3 Proof of Theorem 1.1

By the scaling property of the fractional Brownian motion we see that, as processes indexed by $t \geq 0$,

$$n^{\frac{Hd-1}{2}} \int_0^{nt} f(B(s)) ds \stackrel{\mathcal{L}}{=} n^{\frac{1+Hd}{2}} \int_0^t f(n^H B(s)) ds.$$

Therefore, it suffices to show the theorem for the continuous process

$$F_n(t) := n^{\frac{1+Hd}{2}} \int_0^t f(n^H B(s)) ds.$$

The proof of Theorem 1.1 will be done in two steps. We first show tightness, and then establish the convergence of moments. Tightness will be deduced from the following result.

Proposition 3.1 *For any $0 \leq a < b \leq t$ and any integer $m \geq 1$,*

$$\mathbb{E} \left[(F_n(b) - F_n(a))^{2m} \right] \leq C \left((b-a)^{1-Hd} \int_{\mathbb{R}^{2d}} |f(x)f(y)| |y|^{\frac{1}{H}-d} dx dy \right)^m,$$

where C is a constant depending only on H and m .

Proof. Define

$$D = \{s \in \mathbb{R}^{2m} : a < s_1 < s_2 < \dots < s_{2m} < b\}. \quad (3.1)$$

Using the following identity for $f \in H_0^{\frac{1}{H}-d}$

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{\iota x \cdot \xi} \int_{\mathbb{R}^d} e^{-\iota \xi \cdot y} f(y) dy d\xi,$$

we then have

$$\begin{aligned} & \mathbb{E} \left[(F_n(b) - F_n(a))^{2m} \right] \\ &= (2m)! n^{m(1+Hd)} \mathbb{E} \left(\int_D \prod_{i=1}^{2m} f(n^H B(s_i)) ds \right) \\ &= \frac{(2m)!}{(2\pi)^{2md}} n^{m(1-Hd)} \int_{\mathbb{R}^{2md}} \int_D \prod_{i=1}^{2m} f(y_i) \int_{\mathbb{R}^{2md}} \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^{2m} \xi_i \cdot B(s_i) \right) - \iota \sum_{i=1}^{2m} \frac{y_i \cdot \xi_i}{n^H} \right) d\xi ds dy \\ &= \frac{(2m)!}{(2\pi)^{2md}} n^{m(1-Hd)} \int_{\mathbb{R}^{2md}} \int_D \prod_{i=1}^{2m} f(y_i) \int_{\mathbb{R}^{2md}} \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^{2m} \xi_i \cdot B(s_i) \right) \right) \\ & \quad \times \prod_{i=1}^{2m} \left(\exp \left(-\iota \frac{y_i \cdot \xi_i}{n^H} \right) - 1 \right) d\xi ds dy, \end{aligned} \quad (3.2)$$

where in the last equality we used the fact that $\int_{\mathbb{R}^d} f(x) dx = 0$.

By the local nondeterminism property (2.3), with the convention $s_0 = 0$ and $\eta_{2m+1} = 0$, we can write

$$\begin{aligned}
& \mathbb{E} \left[(F_n(b) - F_n(a))^{2m} \right] \\
& \leq c_1 n^{m(1-Hd)} \int_{\mathbb{R}^{2md}} \int_D \left| \prod_{i=1}^{2m} f(y_i) \right| \int_{\mathbb{R}^{2md}} \exp \left(-\frac{\kappa_H}{2} \sum_{i=1}^{2m} \left| \sum_{j=i}^{2m} \xi_j \right|^2 (s_i - s_{i-1})^{2H} \right) \\
& \quad \times \prod_{i=1}^{2m} \left| e^{-\iota \frac{y_i \cdot \xi_i}{n^H}} - 1 \right| d\xi ds dy \\
& = c_1 n^{m(1-Hd)} \int_{\mathbb{R}^{2md}} \int_D \left| \prod_{i=1}^{2m} f(y_i) \right| \int_{\mathbb{R}^{2md}} \exp \left(-\frac{\kappa_H}{2} \sum_{i=1}^{2m} |\eta_i|^2 (s_i - s_{i-1})^{2H} \right) \\
& \quad \times \prod_{i=1}^{2m} \left| \exp \left(\iota \frac{y_i}{n^H} \cdot (\eta_{i+1} - \eta_i) \right) - 1 \right| d\eta ds dy,
\end{aligned}$$

where we made the change of variables $\eta_i = \sum_{j=i}^{2m} \xi_j$ for $i = 1, \dots, 2m$ in the last equality.

Let $x_i = \eta_i(s_i - s_{i-1})^H$ and $u_i = s_i - s_{i-1}$ for $i = 1, \dots, 2m$. Then

$$\begin{aligned}
& \mathbb{E} \left[(F_n(b) - F_n(a))^{2m} \right] \\
& \leq c_1 n^{m(1-Hd)} \int_{\mathbb{R}^{2md}} \int_{[a,b] \times [0,b-a]^{2m-1}} \prod_{i=1}^{2m} |f(y_i)| \left(\prod_{i=1}^{2m} u_i^{-Hd} \right) \int_{\mathbb{R}^{2md}} \exp \left(-\frac{\kappa_H}{2} \sum_{i=1}^{2m} |x_i|^2 \right) \\
& \quad \times \prod_{i=1}^{2m} \left| \exp \left(\iota \frac{y_i}{n^H} \cdot \left(\frac{x_{i+1}}{u_{i+1}^H} - \frac{x_i}{u_i^H} \right) \right) - 1 \right| dx du dy,
\end{aligned}$$

where $x_{2m+1} = 0$ and the integral on $[a,b] \times [0,b-a]^{2m-1}$ means that $u_1 \in [a,b]$ and $u_i \in [0,b-a]$ for $i = 2, \dots, 2m$.

Let $\sqrt{\kappa_H} X_1, \dots, \sqrt{\kappa_H} X_{2m}$ be independent copies of a d -dimensional standard normal random vector and $X_{2m+1} = 0$. Then the above inequality can be rewritten as

$$\begin{aligned}
& \mathbb{E} \left[(F_n(b) - F_n(a))^{2m} \right] \leq c_2 n^{m(1-Hd)} \int_{\mathbb{R}^{2md}} \int_{[a,b] \times [0,b-a]^{2m-1}} \left(\prod_{i=1}^{2m} |f(y_i)| u_i^{-Hd} \right) \\
& \quad \times \mathbb{E} \left(\prod_{i=1}^{2m} \left| \exp \left(\iota \frac{y_i \cdot X_{i+1}}{n^H u_{i+1}^H} - \iota \frac{y_i \cdot X_i}{n^H u_i^H} \right) - 1 \right| \right) du dy. \tag{3.3}
\end{aligned}$$

Notice that

$$\left| e^{\iota \frac{y_i \cdot (X_{i+1}}{u_{i+1}^H} - \frac{X_i}{u_i^H})}{n^H} - 1 \right| \leq 2 \wedge \left(\left| e^{\iota \frac{y_i \cdot X_{i+1}}{n^H u_{i+1}^H}} - 1 \right| + \left| e^{\iota \frac{y_i \cdot X_i}{n^H u_i^H}} - 1 \right| \right).$$

For each factor in the product inside the expectation in (3.3), we choose the upper bound 2 when i is even and the upper bound

$$\left| e^{\iota \frac{y_i \cdot X_{i+1}}{n^H u_{i+1}^H}} - 1 \right| + \left| e^{\iota \frac{y_i \cdot X_i}{n^H u_i^H}} - 1 \right|$$

when i is odd. Thus, we have

$$\begin{aligned} & \mathbb{E} \left[(F_n(b) - F_n(a))^{2m} \right] \\ & \leq c_3 \int_{\mathbb{R}^{2md}} \int_{[a,b] \times [0,b-a]^{2m-1}} \mathbb{E} \left[\prod_{i=1}^m \left(n^{1-Hd} |f(y_{2i-1})f(y_{2i})| (u_{2i-1}^{-Hd} u_{2i}^{-Hd}) \right. \right. \\ & \quad \left. \left. \times \left(\left| e^{\iota \frac{y_{2i-1} \cdot X_{2i}}{n^H u_{2i}^H}} - 1 \right| + \left| e^{\iota \frac{y_{2i-1} \cdot X_{2i-1}}{n^H u_{2i-1}^H}} - 1 \right| \right) \right) \right] du dy. \end{aligned}$$

Since the above random factors are independent, we have

$$\begin{aligned} & \mathbb{E} \left[(F_n(b) - F_n(a))^{2m} \right] \\ & \leq c_3 \int_{\mathbb{R}^{2md}} \int_{[a,b] \times [0,b-a]^{2m-1}} \prod_{i=1}^m \left(n^{1-Hd} |f(y_{2i-1})f(y_{2i})| (u_{2i-1}^{-Hd} u_{2i}^{-Hd}) \right. \\ & \quad \left. \times \mathbb{E} \left(\left| e^{\iota \frac{y_{2i-1} \cdot X_{2i}}{n^H u_{2i}^H}} - 1 \right| + \left| e^{\iota \frac{y_{2i-1} \cdot X_{2i-1}}{n^H u_{2i-1}^H}} - 1 \right| \right) \right) du dy. \end{aligned}$$

With the change of variables $x = y_{2i-1}$, $y = y_{2i}$, $u = u_{2i}$ and $v = u_{2i-1}$, the above inequality can be rewritten as

$$\begin{aligned} & \mathbb{E} \left[(F_n(b) - F_n(a))^{2m} \right] \\ & \leq c_3 \int_{\mathbb{R}^{2d}} \int_a^b \int_0^{b-a} n^{1-Hd} |f(x)f(y)| (uv)^{-Hd} \mathbb{E} \left(\left| e^{\iota \frac{y \cdot X_2}{n^H u^H}} - 1 \right| + \left| e^{\iota \frac{y \cdot X_1}{n^H v^H}} - 1 \right| \right) du dv dx dy \\ & \quad \times \left(2 \int_{\mathbb{R}^{2d}} \int_0^{b-a} \int_0^{b-a} n^{1-Hd} |f(x)f(y)| (uv)^{-Hd} \mathbb{E} \left(\left| e^{\iota \frac{y \cdot X_1}{n^H u^H}} - 1 \right| \right) du dv dx dy \right)^{m-1}. \quad (3.4) \end{aligned}$$

Notice that $\int_a^b u^{-Hd} du \leq c_4(b-a)^{1-Hd}$. Then, by Lemma 4.2, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^{2d}} \int_a^b \int_0^{b-a} n^{1-Hd} |f(x)f(y)| (uv)^{-Hd} \mathbb{E} \left(\left| e^{\iota \frac{y \cdot X_2}{n^H u^H}} - 1 \right| + \left| e^{\iota \frac{y \cdot X_1}{n^H v^H}} - 1 \right| \right) du dv dx dy \\ & \leq c_5(b-a)^{1-Hd} \int_{\mathbb{R}^{2d}} |f(x)f(y)| |y|^{\frac{1}{H}-d} dx dy \quad (3.5) \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathbb{R}^{2d}} \int_0^{b-a} \int_0^{b-a} n^{1-Hd} |f(x)f(y)| (uv)^{-Hd} \mathbb{E} \left(\left| e^{\iota \frac{y \cdot X_1}{n^H u^H}} - 1 \right| \right) du dv dx dy \\ & \leq c_6(b-a)^{1-Hd} \int_{\mathbb{R}^{2d}} |f(x)f(y)| |y|^{\frac{1}{H}-d} dx dy. \quad (3.6) \end{aligned}$$

Now Proposition 3.1 follows from (3.4), (3.5) and (3.6). \blacksquare

Now we prove that the moments of $F_n(t)$ converge to the corresponding moments of $W(L_t(0))$.

Fix a finite number of disjoint intervals $(a_i, b_i]$ with $i = 1, \dots, N$ and $b_i \leq a_{i+1}$. Let $\mathbf{m} = (m_1, \dots, m_N)$ be a fixed multi-index with $m_i \in \mathbb{N}$ for $i = 1, \dots, N$. Set $\sum_{i=1}^N m_i = |\mathbf{m}|$ and $\prod_{i=1}^N m_i! = \mathbf{m}!$. We need to consider the following sequence of random variables

$$G_n = \prod_{i=1}^N (F_n(b_i) - F_n(a_i))^{m_i}$$

and compute $\lim_{n \rightarrow \infty} \mathbb{E}(G_n)$. Notice that the expectation of G_n can be written as

$$\mathbb{E}(G_n) = \mathbf{m}! n^{\frac{|\mathbf{m}|(1+Hd)}{2}} \mathbb{E} \left(\int_{D_{\mathbf{m}}} \prod_{i=1}^N \prod_{j=1}^{m_i} f(n^H B(s_j^i)) ds \right),$$

where

$$D_{\mathbf{m}} = \{s \in \mathbb{R}^{|\mathbf{m}|} : a_i < s_1^i < \dots < s_{m_i}^i < b_i, 1 \leq i \leq N\}. \quad (3.7)$$

Here and in the sequel we denote the coordinates of a point $s \in \mathbb{R}^{|\mathbf{m}|}$ as $s = (s_j^i)$, where $1 \leq i \leq N$ and $1 \leq j \leq m_i$.

For simplicity of notation, we define

$$J_0 = \{(i, j) : 1 \leq i \leq N, 1 \leq j \leq m_i\}.$$

For any (i_1, j_1) and $(i_2, j_2) \in J_0$, we define the following dictionary ordering

$$(i_1, j_1) \leq (i_2, j_2)$$

if $i_1 < i_2$ or $i_1 = i_2$ and $j_1 \leq j_2$. For any (i, j) in J_0 , under the above ordering, (i, j) is the $(\sum_{k=1}^{i-1} m_k + j)$ -th element in J_0 and we define $\#(i, j) = \sum_{k=1}^{i-1} m_k + j$.

Proposition 3.2 *Suppose that at least one of the exponents m_i is odd. Then*

$$\lim_{n \rightarrow \infty} \mathbb{E}(G_n) = 0.$$

Proof. The proof will be done in several steps.

Step 1 Using similar argument as in (3.2), we obtain

$$\begin{aligned} \mathbb{E}(G_n) &= \frac{\mathbf{m}!}{(2\pi)^{|\mathbf{m}|d}} n^{\frac{|\mathbf{m}|(1-Hd)}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \left(\prod_{i=1}^N \prod_{j=1}^{m_i} f(y_j^i) \right) \\ &\quad \times \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^N \sum_{j=1}^{m_i} \xi_j^i \cdot B(s_j^i) \right) - \iota \sum_{i=1}^N \sum_{j=1}^{m_i} \frac{y_j^i \cdot \xi_j^i}{n^H} \right) d\xi ds dy \\ &= \frac{\mathbf{m}!}{(2\pi)^{|\mathbf{m}|d}} n^{\frac{|\mathbf{m}|(1-Hd)}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \left(\prod_{i=1}^N \prod_{j=1}^{m_i} f(y_j^i) \right) \\ &\quad \times \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^N \sum_{j=1}^{m_i} \xi_j^i \cdot B(s_j^i) \right) \right) \prod_{i=1}^N \prod_{j=1}^{m_i} \left(e^{-\iota \frac{y_j^i \cdot \xi_j^i}{n^H}} - 1 \right) d\xi ds dy, \end{aligned}$$

where we used the fact $\int_{\mathbb{R}^d} f(x) dx = 0$ in the last equality.

By the local nondeterminism property (2.3), with the convention $s_0^i = s_{m_{i-1}}^{i-1}$ for $2 \leq i \leq N$ and $s_0^1 = 0$,

$$\text{Var} \left(\sum_{i=1}^N \sum_{j=1}^{m_i} \xi_j^i \cdot B(s_j^i) \right) \geq \kappa_H \sum_{i=1}^N \sum_{j=1}^{m_i} \left| \sum_{(l,k) \geq (i,j)} \xi_k^l \right|^2 (s_j^i - s_{j-1}^i)^{2H}.$$

Let $F(y) = \prod_{i=1}^N \prod_{j=1}^{m_i} f(y_j^i)$ and make the change of variables $\eta_j^i = \sum_{(l,k) \geq (i,j)} \xi_k^l$ for $1 \leq i \leq N$ and $1 \leq j \leq m_i$. Then we can estimate $E(G_n)$ as follows:

$$\begin{aligned} |\mathbb{E}(G_n)| &\leq c_1 n^{\frac{|\mathbf{m}|(1-Hd)}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}} \int_{\mathbb{R}^{|\mathbf{m}|d}} |F(y)| \exp \left(-\frac{\kappa_H}{2} \sum_{i=1}^N \sum_{j=1}^{m_i} |\eta_j^i|^2 (s_j^i - s_{j-1}^i)^{2H} \right) \\ &\quad \times \left(\prod_{i=1}^N \prod_{j=1}^{m_i} \left| \exp \left(-\iota \frac{y_j^i}{n^H} \cdot (\eta_j^i - \eta_{j+1}^i) \right) - 1 \right| \right) d\eta ds dy \\ &= c_1 n^{\frac{|\mathbf{m}|(1-Hd)}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}} |F(y)| \left(\prod_{i=1}^N \prod_{j=1}^{m_i} (s_j^i - s_{j-1}^i)^{-Hd} \right) \\ &\quad \times \mathbb{E} \left(\prod_{i=1}^N \prod_{j=1}^{m_i} \left| \exp \left(\iota \frac{y_j^i \cdot X_{j+1}^i}{n^H (s_{j+1}^i - s_j^i)^H} - \iota \frac{y_j^i \cdot X_j^i}{n^H (s_j^i - s_{j-1}^i)^H} \right) - 1 \right| \right) ds dy, \quad (3.8) \end{aligned}$$

where $X_{m_N+1}^N = 0$, $X_{m_i+1}^i = X_1^{i+1}$ for $1 \leq i \leq N-1$, and $\sqrt{\kappa_H} X_j^i$ ($1 \leq i \leq N$, $1 \leq j \leq m_i$) are independent copies of a d -dimensional standard normal random vector.

Denote the expectation in (3.8) by I . That is,

$$I = \mathbb{E} \left(\prod_{i=1}^N \prod_{j=1}^{m_i} \left| \exp \left(\iota \frac{y_j^i \cdot X_{j+1}^i}{n^H (s_{j+1}^i - s_j^i)^H} - \iota \frac{y_j^i \cdot X_j^i}{n^H (s_j^i - s_{j-1}^i)^H} \right) - 1 \right| \right) = \mathbb{E} \left(\prod_{(i,j) \in J_0} I_{i,j} \right),$$

where

$$I_{i,j} = \left| \exp \left(\iota \frac{y_j^i \cdot X_{j+1}^i}{n^H (s_{j+1}^i - s_j^i)^H} - \iota \frac{y_j^i \cdot X_j^i}{n^H (s_j^i - s_{j-1}^i)^H} \right) - 1 \right|$$

for $1 \leq i \leq N$ and $1 \leq j \leq m_i$.

Notice that the random variables $I_{i,j}$ for $(i,j) \in J_0$ are dependent. We are going to choose a proper subset of J_0 in the following way. Assume that m_ℓ is the first odd exponent. Then we choose all the factors $I_{i,j}$ such that $\#(i,j) < \#(\ell, m_\ell)$ and $\#(i,j)$ is odd. Then, we choose all the factors $I_{i,j}$ such that $\#(i,j) > \#(\ell, m_\ell) + 1$ and $\#(i,j)$ is even. Notice that all these factors are mutually independent and they are also independent of the product $I_{\ell, m_\ell} I_{\ell+1, 1}$. The lack of independence of the two factors I_{ℓ, m_ℓ} and $I_{\ell+1, 1}$ will be compensated by the fact that the integral of $(s_1^{\ell+1} - s_{m_\ell}^\ell)^{-\beta}$ is finite for any $\beta < 2$, because we have the constraint $s_{m_\ell}^\ell < b_\ell < s_1^{\ell+1}$. To make this argument more precise, let us define

$$J_\ell = J_{\ell,1} \cup J_{\ell,2},$$

where

$$J_{\ell,1} = \{(i,j) \in J_0 : \#(i,j) < \#(\ell, m_\ell) \text{ and } \#(i,j) \text{ odd}\}$$

and

$$J_{\ell,2} = \{(i, j) \in J_0 : \#(i, j) > \#(\ell, m_\ell) + 1 \text{ and } \#(i, j) \text{ even}\}.$$

Notice that $I_{i,j} \leq 2$ for all $(i, j) \in J_0$. Then

$$I \leq \begin{cases} c_2 \mathbb{E} \left(I_{\ell, m_\ell} I_{\ell+1, 1} \prod_{(i, j) \in J_\ell} I_{i, j} \right) & \text{if } \ell \neq N, \\ c_2 \mathbb{E} \left(I_{\ell, m_\ell} \prod_{(i, j) \in J_\ell} I_{i, j} \right) & \text{if } \ell = N. \end{cases}$$

Step 2 We first consider the case $\ell \neq N$. In this case, the number of elements in J_ℓ is $\lceil \frac{|\mathbf{m}|}{2} \rceil - 1$ and

$$\begin{aligned} |\mathbb{E}(G_n)| &\leq c_3 n^{\frac{|\mathbf{m}|(1-Hd)}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}} |F(y)| \left(\prod_{(i, j) \in J_0} (s_j^i - s_{j-1}^i)^{-Hd} \right) \\ &\quad \times \mathbb{E} \left(I_{\ell, m_\ell} I_{\ell+1, 1} \right) \prod_{(i, j) \in J_\ell} \mathbb{E} (I_{i, j}) ds dy. \end{aligned}$$

In the last inequality, we used the fact that all random variables $I_{\ell, m_\ell} I_{\ell+1, 1}$ and $I_{i, j}$ for $(i, j) \in J_\ell$ are independent.

Since $|e^{\iota(z_1 - z_2)} - 1| \leq |e^{\iota z_1} - 1| + |e^{\iota z_2} - 1|$ for all $z_1, z_2 \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E} (I_{\ell, m_\ell} I_{\ell+1, 1}) &\leq \mathbb{E} \left\{ \left(|e^{\iota \frac{y_{m_\ell}^\ell \cdot X_1^{\ell+1}}{n^H (s_1^{\ell+1} - s_{m_\ell}^\ell)^H}} - 1| + |e^{\iota \frac{y_{m_\ell}^\ell \cdot X_{m_\ell}^\ell}{n^H (s_{m_\ell}^\ell - s_{m_\ell-1}^\ell)^H}} - 1| \right) \right. \\ &\quad \left. \times \left(|e^{\iota \frac{y_1^{\ell+1} \cdot X_2^{\ell+1}}{n^H (s_2^{\ell+1} - s_1^{\ell+1})^H}} - 1| + |e^{\iota \frac{y_1^{\ell+1} \cdot X_1^{\ell+1}}{n^H (s_1^{\ell+1} - s_{m_\ell}^\ell)^H}} - 1| \right) \right\}. \end{aligned}$$

Notice that $X_2^{\ell+1}, X_1^{\ell+1}$ and $X_{m_\ell}^\ell$ are independent. As a consequence, we can write

$$\mathbb{E} (I_{\ell, m_\ell} I_{\ell+1, 1}) \leq I_1^\ell + I_2^\ell,$$

where

$$\begin{aligned} I_1^\ell &= \mathbb{E} \left| e^{\iota \frac{y_{m_\ell}^\ell \cdot X_1^{\ell+1}}{n^H (s_1^{\ell+1} - s_{m_\ell}^\ell)^H}} - 1 \right| \mathbb{E} \left| e^{\iota \frac{y_1^{\ell+1} \cdot X_2^{\ell+1}}{n^H (s_2^{\ell+1} - s_1^{\ell+1})^H}} - 1 \right| \\ &\quad + \mathbb{E} \left| e^{\iota \frac{y_{m_\ell}^\ell \cdot X_{m_\ell}^\ell}{n^H (s_{m_\ell}^\ell - s_{m_\ell-1}^\ell)^H}} - 1 \right| \mathbb{E} \left| e^{\iota \frac{y_1^{\ell+1} \cdot X_2^{\ell+1}}{n^H (s_2^{\ell+1} - s_1^{\ell+1})^H}} - 1 \right| \\ &\quad + \mathbb{E} \left| e^{\iota \frac{y_{m_\ell}^\ell \cdot X_{m_\ell}^\ell}{n^H (s_{m_\ell}^\ell - s_{m_\ell-1}^\ell)^H}} - 1 \right| \mathbb{E} \left| e^{\iota \frac{y_1^{\ell+1} \cdot X_1^{\ell+1}}{n^H (s_1^{\ell+1} - s_{m_\ell}^\ell)^H}} - 1 \right| \end{aligned}$$

and

$$I_2^\ell = \mathbb{E} \left(\left| e^{\iota \frac{y_{m_\ell}^\ell \cdot X_1^{\ell+1}}{n^H (s_1^{\ell+1} - s_{m_\ell}^\ell)^H}} - 1 \right| \left| e^{\iota \frac{y_1^{\ell+1} \cdot X_1^{\ell+1}}{n^H (s_1^{\ell+1} - s_{m_\ell}^\ell)^H}} - 1 \right| \right). \quad (3.9)$$

Therefore,

$$|\mathbb{E}(G_n)| \leq c_3 n^{\frac{|\mathbf{m}|(1-Hd)}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} |F(y)| (G_1 + G_2) dy, \quad (3.10)$$

where

$$G_{k,n} = \int_{D_{\mathbf{m}}} \left(\prod_{(i,j) \in J_0} (s_j^i - s_{j-1}^i)^{-Hd} \right) I_k^\ell \prod_{(i,j) \in J_\ell} \mathbb{E}(I_{i,j}) ds$$

for $k = 1, 2$.

We claim that

$$G_{1,n} \leq c_4 n^{([\frac{|\mathbf{m}|}{2}] + 1)(Hd-1)} |y_1^{\ell+1}|^{\frac{1}{H}-d} |y_{m_\ell}^\ell|^{\frac{1}{H}-d} \prod_{(i,j) \in J_\ell} |y_j^i|^{\frac{1}{H}-d}.$$

In fact, making the change of variables $v_j^i = s_j^i - s_{j-1}^i$ for all $(i, j) \in J_0$, and defining

$$a_{i,j} = \begin{cases} (v_j^i v_{j+1}^i)^{-Hd} \mathbb{E}(I_{i,j}) & \text{if } (i, j) \in J_\ell \text{ and } (i, j) \neq (N, m_N); \\ (v_{m_N}^N)^{-Hd} \mathbb{E}(I_{N, m_N}) & \text{if } (i, j) \in J_\ell \text{ and } (i, j) = (N, m_N), \end{cases}$$

and $a_1^\ell = (v_2^{\ell+1} v_1^{\ell+1} v_{m_\ell}^\ell)^{-Hd} I_1^\ell$, we obtain

$$\begin{aligned} G_{1,n} &\leq \int_{[0, b_N]^{|\mathbf{m}|}} a_1^\ell \prod_{(i,j) \in J_\ell} a_{i,j} dv \\ &= \int_{[0, b_N]^3} a_1^\ell dv_2^{\ell+1} dv_1^{\ell+1} dv_{m_\ell}^\ell \prod_{(i,j) \in J_\ell} \int_{[0, b_N]^2} a_{i,j} dv_j^i dv_{j+1}^i \\ &\leq c_5 n^{([\frac{|\mathbf{m}|}{2}] + 1)(Hd-1)} |y_1^{\ell+1}|^{\frac{1}{H}-d} |y_{m_\ell}^\ell|^{\frac{1}{H}-d} \prod_{(i,j) \in J_\ell} |y_j^i|^{\frac{1}{H}-d}. \end{aligned} \quad (3.11)$$

Here we used Lemma 4.2 in the last inequality $[\frac{|\mathbf{m}|}{2}] + 1$ times.

For any $\beta \in [0, 1]$, we have $|e^{\iota z} - 1| \leq c_\beta |z|^\beta$ for all $z \in \mathbb{R}$. Recall the definition of I_2^ℓ in (3.9). We then have

$$\begin{aligned} I_2^\ell &\leq c_6 n^{-2H\beta} |y_{m_\ell}^\ell|^\beta |y_1^{\ell+1}|^\beta (s_1^{\ell+1} - s_{m_\ell}^\ell)^{-2H\beta} \mathbb{E} |X_1^{\ell+1}|^{2\beta} \\ &\leq c_7 n^{-2H\beta} |y_{m_\ell}^\ell|^\beta |y_1^{\ell+1}|^\beta (s_1^{\ell+1} - s_{m_\ell}^\ell)^{-2H\beta}. \end{aligned}$$

So

$$\begin{aligned} G_{2,n} &\leq c_7 n^{-2H\beta} |y_{m_\ell}^\ell|^\beta |y_1^{\ell+1}|^\beta \\ &\quad \times \int_{D_{\mathbf{m}}} (s_1^{\ell+1} - s_{m_\ell}^\ell)^{-2H\beta} \left(\prod_{(i,j) \in J_0} (s_j^i - s_{j-1}^i)^{-Hd} \right) I_2^\ell \prod_{(i,j) \in J_\ell} \mathbb{E}(I_{i,j}) ds. \end{aligned}$$

Define $J_{\ell,3} = \{(i, j) \in J_0 : \#(i, j) < \#(\ell, m_\ell)\}$ and

$$D_{\mathbf{m}}^\ell = \{a_i < s_1^i < \dots < s_{m_i}^i < b_i, 1 \leq i \leq \ell - 1; a_\ell < s_1^\ell < \dots < s_{m_\ell}^\ell < b_\ell\}.$$

Integrating the above integral with respect to s_j^i for $(i, j) \in J_{\ell,2}$ and using Lemma 4.2,

$$\begin{aligned} G_{2,n} &\leq c_8 n^{-2H\beta} n^{\#J_{\ell,2}(Hd-1)} |y_{m_\ell}^\ell|^\beta |y_1^{\ell+1}|^\beta \prod_{(i,j) \in J_{\ell,2}} |y_j^i|^{\frac{1}{H}-d} \\ &\quad \times \int_{D_{\mathbf{m}}^\ell} I_3^\ell \prod_{(i,j) \in J_{\ell,3}} (s_j^i - s_{j-1}^i)^{-Hd} \prod_{(i,j) \in J_{\ell,1}} \mathbb{E}(I_{i,j}) ds, \end{aligned}$$

where $\#J_{\ell,2}$ is the cardinality of $J_{\ell,2}$ and

$$I_3^\ell = \int_{s_{m_\ell-1}^\ell}^{b_l} \int_{a_{\ell+1}}^{b_{l+1}} \int_{s_1^{\ell+1}}^{b_{l+1}} (s_2^{\ell+1} - s_1^{\ell+1})^{-Hd} (s_1^{\ell+1} - s_{m_\ell}^\ell)^{-Hd-2H\beta} (s_{m_\ell}^\ell - s_{m_\ell-1}^\ell)^{-Hd} ds_2^{\ell+1} ds_1^{\ell+1} ds_{m_\ell}^\ell.$$

We observe that, if $1 - Hd < 2H\beta \leq 2 - 2Hd$,

$$\begin{aligned} I_3^\ell &\leq c_9 \int_{s_{m_\ell-1}^\ell}^{b_l} \int_{a_{\ell+1}}^{b_{l+1}} (s_1^{\ell+1} - s_{m_\ell}^\ell)^{-Hd-2H\beta} (s_{m_\ell}^\ell - s_{m_\ell-1}^\ell)^{-Hd} ds_1^{\ell+1} ds_{m_\ell}^\ell \\ &\leq c_{10} \int_{s_{m_\ell-1}^\ell}^{b_l} (a_{\ell+1} - s_{m_\ell}^\ell)^{1-Hd-2H\beta} (s_{m_\ell}^\ell - s_{m_\ell-1}^\ell)^{-Hd} ds_{m_\ell}^\ell \\ &\leq c_{11} (a_{\ell+1} - a_\ell)^{2-2Hd-2H\beta}. \end{aligned}$$

As a consequence,

$$\begin{aligned} G_{2,n} &\leq c_{12} n^{-2H\beta} n^{\#J_{\ell,2}(Hd-1)} |y_{m_\ell}^\ell|^\beta |y_1^{\ell+1}|^\beta \prod_{(i,j) \in J_{\ell,2}} |y_j^i|^{\frac{1}{H}-d} \\ &\quad \times \int_{D_{\mathbf{m}}^\ell} \prod_{(i,j) \in J_{\ell,3}} (s_j^i - s_{j-1}^i)^{-Hd} \prod_{(i,j) \in J_{\ell,1}} \mathbb{E}(I_{i,j}) ds \\ &\leq c_{13} n^{-2H\beta} n^{\#J_\ell(Hd-1)} |y_{m_\ell}^\ell|^\beta |y_1^{\ell+1}|^\beta \prod_{(i,j) \in J_\ell} |y_j^i|^{\frac{1}{H}-d} \\ &= c_{13} n^{-2H\beta} n^{(\lfloor \frac{|\mathbf{m}|}{2} \rfloor - 1)(Hd-1)} |y_{m_\ell}^\ell|^\beta |y_1^{\ell+1}|^\beta \prod_{(i,j) \in J_\ell} |y_j^i|^{\frac{1}{H}-d}. \end{aligned} \quad (3.12)$$

Choosing $\beta = \frac{1}{H} - d$ in (3.12),

$$G_{2,n} \leq c_{13} n^{(\lfloor \frac{|\mathbf{m}|}{2} \rfloor + 1)(Hd-1)} |y_{m_\ell}^\ell|^{\frac{1}{H}-d} |y_1^{\ell+1}|^{\frac{1}{H}-d} \prod_{(i,j) \in J_\ell} |y_j^i|^{\frac{1}{H}-d}. \quad (3.13)$$

Substituting (3.11) and (3.13) into (3.10) yields,

$$|\mathbb{E}(G_n)| \leq c_{14} n^{-\frac{1-Hd}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} |F(y)| |y_{m_\ell}^\ell|^{\frac{1}{H}-d} |y_1^{\ell+1}|^{\frac{1}{H}-d} \prod_{(i,j) \in J_\ell} |y_j^i|^{\frac{1}{H}-d} dy. \quad (3.14)$$

Step 3 Now we consider the case $\ell = N$. In this case, $J_\ell = J_{\ell,1}$ and

$$\begin{aligned} |\mathbb{E}(G_n)| &\leq c_{15} n^{\frac{|\mathbf{m}|(1-Hd)}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}} |F(y)| \prod_{(i,j) \in J_0} (s_j^i - s_{j-1}^i)^{-Hd} \\ &\quad \times \mathbb{E}(I_{N,m_N}) \prod_{(i,j) \in J_{N,1}} \mathbb{E}(I_{i,j}) ds dy. \end{aligned}$$

Define $J_{N,3} = \{(i,j) \in J_0 : \#(i,j) < \#(N, m_N)\}$ and

$$D_{\mathbf{m}}^N = \{a_i < s_1^i < \dots < s_{m_i}^i < b_i, 1 \leq i \leq N-1; a_\ell < s_1^N < \dots < s_{m_{N-1}}^N < b_N\}.$$

Integrating the above integral with respect to $s_{m_N}^N$ and using Lemma 4.2 yield,

$$\begin{aligned} |\mathbb{E}(G_n)| &\leq c_{16} n^{\frac{(|\mathbf{m}|-2)(1-Hd)}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}^N} |F(y)| |y_{m_N}^N|^{\frac{1}{H}-d} \\ &\quad \times \prod_{(i,j) \in J_{N,3}} (s_j^i - s_{j-1}^i)^{-Hd} \prod_{(i,j) \in J_{N,1}} \mathbb{E}(I_{i,j}) ds dy. \end{aligned}$$

Using similar arguments as in **Step 2**,

$$|\mathbb{E}(G_n)| \leq c_{17} n^{-\frac{1-Hd}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} |F(y)| |y_{m_N}^N|^{\frac{1}{H}-d} \prod_{(i,j) \in J_{N,1}} |y_j^i|^{\frac{1}{H}-d} dy. \quad (3.15)$$

Step 4 Recall that $f \in H_0^{\frac{1}{H}-d}$. Then, from (3.14) and (3.15), we see that $|\mathbb{E}(G_n)|$ is bounded by a multiple of $n^{-\frac{1-Hd}{2}}$. Our result now follows from taking the limit. \blacksquare

In the sequel, we consider the convergence of moments when all exponents m_i are even. Recall the definition of $D_{\mathbf{m}}$ in (3.7). For $1 \leq \ell \leq N$ and $1 \leq k \leq \frac{m_\ell}{2}$, we define

$$O_k^\ell = D_{\mathbf{m}} \cap \left\{ \frac{s_{2k}^\ell - s_{2k-2}^\ell}{2} < s_{2k}^\ell - s_{2k-1}^\ell \right\}.$$

The following result tells us that the integrals over the domain O_k^ℓ do not contribute to the limit of the moments. This result will play a fundamental role in computing the limits of even moments.

Proposition 3.3 *For any $1 \leq \ell \leq N$ and $1 \leq k \leq \frac{m_\ell}{2}$,*

$$\lim_{n \rightarrow \infty} n^{\frac{|\mathbf{m}|(1+Hd)}{2}} \mathbb{E} \left(\int_{O_k^\ell} \prod_{i=1}^N \prod_{j=1}^{m_i} f(n^H B(s_j^i)) ds \right) = 0.$$

Proof. Using the arguments and notation in the proof of Proposition 3.2, we obtain

$$\begin{aligned} &n^{\frac{|\mathbf{m}|(1+Hd)}{2}} \left| \mathbb{E} \left(\int_{O_k^\ell} \prod_{i=1}^N \prod_{j=1}^{m_i} f(n^H B(s_j^i)) ds \right) \right| \\ &\leq c_1 n^{\frac{|\mathbf{m}|(1-Hd)}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{O_k^\ell} |F(y)| \left(\prod_{i=1}^N \prod_{j=1}^{m_i} (s_j^i - s_{j-1}^i)^{-H} \right) \\ &\quad \times \mathbb{E} \left(\prod_{i=1}^N \prod_{j=1}^{m_i} \left| \exp \left(\iota \frac{y_j^i \cdot X_{j+1}^i}{n^H (s_{j+1}^i - s_j^i)^H} - \iota \frac{y_j^i \cdot X_j^i}{n^H (s_j^i - s_{j-1}^i)^H} \right) - 1 \right| \right) ds dy, \end{aligned}$$

where $X_{m_N+1}^N = 0$, $X_{m_i+1}^i = X_1^{i+1}$ for $1 \leq i \leq N-1$, and $\sqrt{\kappa_H} X_j^i$ ($1 \leq i \leq N$, $1 \leq j \leq m_i$) are independent copies of a d -dimensional standard normal random vector.

We make the change of variables $v_j^i = s_j^i - s_{j-1}^i$ for all $(i, j) \in J_0$. The integral domain O_k^ℓ becomes

$$D_k^\ell = \left\{ v \in \mathbb{R}_+^{|\mathbf{m}|} : a_1 < \sum_{(i,j) \in J_0} v_j^i < b_N, v_{2k-1}^\ell < v_{2k}^\ell \right\}.$$

For $(i, j) \in J_0$, define

$$I_{i,j} = \left| \exp \left(\iota \frac{y_j^i \cdot X_{j+1}^i}{n^H (v_{j+1}^i)^H} - \iota \frac{y_j^i \cdot X_j^i}{n^H (v_j^i)^H} \right) - 1 \right|.$$

Then

$$\begin{aligned} & n^{\frac{|\mathbf{m}|(1+Hd)}{2}} \left| \mathbb{E} \left(\int_{O_k^\ell} \prod_{i=1}^N \prod_{j=1}^{m_i} f(n^H B(s_j^i)) ds \right) \right| \\ & \leq c_1 n^{\frac{|\mathbf{m}|(1-Hd)}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_k^\ell} F(y) \mathbb{E} \left(\prod_{(i,j) \in J_0} (v_j^i)^{-Hd} I_{i,j} \right) dv dy. \end{aligned} \quad (3.16)$$

Next we estimate the expectation in (3.16). We are going to use an argument similar to the one used in the proof of Proposition 3.2, based on the selection of some factors in the above product. Here, the dependent product that will play a basic role will be $I_{\ell,2k} I_{\ell,2k-1}$, due to the definition of the set O_k^ℓ . Define

$$J_k^\ell = J_{k,1}^\ell \cup J_{k,2}^\ell,$$

where

$$\begin{aligned} J_{k,1}^\ell &= \{(i, j) \in J_0 : \#(i, j) < \#(\ell, 2k - 2), \#(i, j) \text{ odd}\}, \\ J_{k,2}^\ell &= \{(i, j) \in J_0 : \#(i, j) > \#(\ell, 2k), \#(i, j) \text{ even}\}. \end{aligned}$$

Since all exponents m_i are even, the number of elements in J_k^ℓ is $\frac{|\mathbf{m}|-2}{2} = \frac{|\mathbf{m}|}{2} - 1$. From the definition of $I_{i,j}$, we know that random variables $I_{\ell,2k} I_{\ell,2k-1}$ and $I_{i,j}$ for $(i, j) \in J_k^\ell$ are independent. Then

$$\begin{aligned} & n^{\frac{|\mathbf{m}|(1+Hd)}{2}} \left| \mathbb{E} \left(\int_{O_k^\ell} \prod_{i=1}^N \prod_{j=1}^{m_i} f(n^H B(s_j^i)) ds \right) \right| \\ & \leq c_2 n^{\frac{|\mathbf{m}|(1-Hd)}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_k^\ell} F(y) \mathbb{E}(I_{\ell,2k} I_{\ell,2k-1}) \prod_{(i,j) \in J_0} (v_j^i)^{-Hd} \prod_{(i,j) \in J_k^\ell} \mathbb{E}(I_{i,j}) dv dy. \end{aligned}$$

For $(i, j) \in J_k^\ell$ and $(i, j) \neq (N, m_N)$, define

$$a_{i,j} = (v_j^i v_{j+1}^i)^{-Hd} \mathbb{E}(I_{i,j})$$

and $a_{N,m_N} = (v_{m_N}^N)^{-Hd} \mathbb{E}(I_{N,m_N})$. From Lemma 4.2, we obtain

$$\int_{[0,b_N]^2} a_{i,j} dv_j^i dv_{j+1}^i \leq c_3 n^{Hd-1} |y_j^i|^{\frac{1}{H}-d}$$

for all $(i, j) \in J_k^\ell$. Therefore,

$$n^{\frac{|\mathbf{m}|(1+Hd)}{2}} \left| \mathbb{E} \left(\int_{O_k^\ell} \prod_{i=1}^N \prod_{j=1}^{m_i} f(n^H B(s_j^i)) ds \right) \right| \leq c_2 n^{1-Hd} \int_{\mathbb{R}^{|\mathbf{m}|d}} F(y) \prod_{(i,j) \in J_k^\ell} |y_j^i|^{\frac{1}{H}-d} I_k^\ell dy, \quad (3.17)$$

where

$$I_k^\ell = \int_0^{b_N} \int_{v_{2k-1}^\ell}^{b_N} \int_0^{b_N} (v_{2k+1}^\ell v_{2k}^\ell v_{2k-1}^\ell)^{-Hd} \mathbb{E}(I_{\ell,2k} I_{\ell,2k-1}) dv_{2k+1}^\ell dv_{2k}^\ell dv_{2k-1}^\ell.$$

Notice that $|e^{\iota(z_1-z_2)} - 1| \leq |e^{\iota z_1} - 1| + |e^{\iota z_2} - 1|$ for all $z_1, z_2 \in \mathbb{R}$. Using the independence of $X_{2k-1}^\ell, X_{2k}^\ell$ and X_{2k+1}^ℓ , we obtain

$$\mathbb{E}(I_{\ell,2k-1} I_{\ell,2k}) \leq A_{k,1}^\ell + A_{k,2}^\ell,$$

where

$$\begin{aligned} A_{k,1}^\ell &= \mathbb{E} |e^{\iota \frac{y_{2k-1}^\ell \cdot X_{2k}^\ell}{n^H (v_{2k}^\ell)^H}} - 1| \mathbb{E} |e^{\iota \frac{y_{2k}^\ell \cdot X_{2k+1}^\ell}{n^H (v_{2k+1}^\ell)^H}} - 1| + \mathbb{E} |e^{\iota \frac{y_{2k-1}^\ell \cdot X_{2k-1}^\ell}{n^H (v_{2k-1}^\ell)^H}} - 1| \mathbb{E} |e^{\iota \frac{y_{2k}^\ell \cdot X_{2k+1}^\ell}{n^H (v_{2k+1}^\ell)^H}} - 1| \\ &\quad + \mathbb{E} |e^{\iota \frac{y_{2k-1}^\ell \cdot X_{2k-1}^\ell}{n^H (v_{2k-1}^\ell)^H}} - 1| \mathbb{E} |e^{\iota \frac{y_{2k}^\ell \cdot X_{2k}^\ell}{n^H (v_{2k}^\ell)^H}} - 1| \end{aligned}$$

and

$$A_{k,2}^\ell = \mathbb{E} (|e^{\iota \frac{y_{2k-1}^\ell \cdot X_{2k}^\ell}{n^H (v_{2k}^\ell)^H}} - 1| |e^{\iota \frac{y_{2k}^\ell \cdot X_{2k}^\ell}{n^H (v_{2k}^\ell)^H}} - 1|).$$

Now we have

$$I_k^\ell = I_{k,1}^\ell + I_{k,2}^\ell, \quad (3.18)$$

where

$$I_{k,i}^\ell = \int_0^{b_N} \int_{v_{2k-1}^\ell}^{b_N} \int_0^{b_N} (v_{2k+1}^\ell v_{2k}^\ell v_{2k-1}^\ell)^{-Hd} A_{k,i}^\ell dv_{2k+1}^\ell dv_{2k}^\ell dv_{2k-1}^\ell$$

for $i = 1, 2$. By Lemma 4.2,

$$I_{k,1}^\ell \leq c_3 n^{-2(1-Hd)} |y_{2k-1}^\ell|^{\frac{1}{H}-d} |y_{2k}^\ell|^{\frac{1}{H}-d}. \quad (3.19)$$

For any $\beta \in [0, 1]$, we have $|e^{\iota z} - 1| \leq c_\beta |z|^\beta$ for all $z \in \mathbb{R}$. Then,

$$A_{k,2}^\ell \leq c_4 n^{-2H\beta} (v_{2k}^\ell)^{-2H\beta} |y_{2k-1}^\ell|^\beta |y_{2k}^\ell|^\beta.$$

Therefore, if $1 - Hd < 2H\beta < 2 - 2Hd$,

$$\begin{aligned} I_{k,2}^\ell &\leq c_5 n^{-2H\beta} |y_{2k-1}^\ell|^\beta |y_{2k}^\ell|^\beta \int_0^{b_N} \int_{v_{2k-1}^\ell}^{b_N} \int_0^{b_N} \\ &\quad (v_{2k+1}^\ell)^{-Hd} (v_{2k}^\ell)^{-Hd-2H\beta} (v_{2k-1}^\ell)^{-Hd} dv_{2k+1}^\ell dv_{2k}^\ell dv_{2k-1}^\ell \\ &\leq c_6 n^{-2H\beta} |y_{2k-1}^\ell|^\beta |y_{2k}^\ell|^\beta \int_0^{b_N} (v_{2k-1}^\ell)^{1-2Hd-2H\beta} dv_{2k-1}^\ell \\ &\leq c_7 n^{-2H\beta} |y_{2k-1}^\ell|^\beta |y_{2k}^\ell|^\beta. \end{aligned}$$

Choose $\beta = \frac{3(1-Hd)}{4H}$,

$$I_{k,2}^\ell \leq c_7 n^{-\frac{3(1-Hd)}{2}} |y_{2k-1}^\ell|^{\frac{3(1-Hd)}{4H}} |y_{2k}^\ell|^{\frac{3(1-Hd)}{4H}}. \quad (3.20)$$

Substituting (3.19) and (3.20) into (3.18), we obtain

$$I_k^\ell \leq c_8 n^{-\frac{3(1-Hd)}{2}} \left(|y_{2k-1}^\ell| \frac{3(1-Hd)}{4H} |y_{2k}^\ell| \frac{3(1-Hd)}{4H} + |y_{2k-1}^\ell|^{\frac{1}{H}-d} |y_{2k}^\ell|^{\frac{1}{H}-d} \right). \quad (3.21)$$

Our result now follows easily from (3.17), (3.21) and the assumption $f \in H_0^{\frac{1}{H}-d}$. \blacksquare

Before proceeding with the proof of the convergence of moments when all exponents m_i are even, we would like to make some heuristic remarks in the simple case $\mathbb{E}(F_n(t)^2)$, that might help the reader to understand our technical computations. We can write

$$\mathbb{E}(F_n(t)^2) = 2n^{1+Hd} \int_0^t \int_0^{s_2} \mathbb{E}(f(n^H B(s_1)) f(n^H B(s_2))) ds_1 ds_2.$$

Making the change of variables $u_1 = n(s_2 - s_1)$ and $u_2 = s_2$ yields

$$\begin{aligned} \mathbb{E}(F_n(t)^2) &= 2n^{Hd} \int_0^t \int_0^{nu_2} \mathbb{E}\left(f(n^H B(u_2 - \frac{u_1}{n})) f(n^H B(u_2))\right) du_1 du_2 \\ &= 2n^{Hd} \int_0^t \int_0^{nu_2} \int_{\mathbb{R}^{2d}} f(n^H x + y) f(n^H x) p_n(x, y) dx dy du_1 du_2 \\ &= 2 \int_0^t \int_0^{nu_2} \int_{\mathbb{R}^{2d}} f(x + y) f(x) p_n(\frac{x}{n^H}, y) dx dy du_1 du_2, \end{aligned} \quad (3.22)$$

where p_n is the density of the $2d$ -dimensional random vector $(B(u_2), n^H(B(u_2 - \frac{u_1}{n}) - B(u_2)))$. Then, as n tends to infinity, $u_1^{-H} n^H (B(u_2 - \frac{u_1}{n}) - B(u_2))$ converges in law to a d -dimensional standard normal random vector independent of B . So, formally we obtain that the expectation $\mathbb{E}(F_n(t)^2)$ converges to

$$\frac{2}{(2\pi)^{\frac{d}{2}}} \mathbb{E}\left(\int_0^t \delta(B(u_2)) du_2\right) \int_0^\infty \int_{\mathbb{R}^{2d}} f(x + y) f(x) u_1^{-Hd} \left(e^{-\frac{|y|^2}{2u_1^{2H}}} - 1\right) dy dx du_1,$$

which is equal to $C_{H,d} \|f\|_{\frac{1}{H}-d}^2 \mathbb{E}(W(L_t(0))^2)$. Notice also that we have been able to add the term -1 because the integral of f is zero. In conclusion, the term $B(u_2)$ appearing in (3.22) contributes to the local time at zero whereas $n^H(B(u_2 - \frac{u_1}{n}) - B(u_2))$ becomes independent of B in the limit and its expectation contributes to the constant $C_{H,d}$. When computing an even moment, for each couple of consecutive factors we will observe this phenomenon.

The main difficulty to make this argument rigorous is to compute the limit of the integral of the density p_n over the interval $[0, nu_2]$. To overcome this difficulty we will first integrate on a compact $[0, K]$, and then show that the integral over $[K, nu_2]$ converges to zero as K tends to infinity, uniformly in n . However, this convergence holds only if we integrate over $[K, \frac{nu_2}{2}]$, which is fine because Proposition 3.3 implies that the integral over $[\frac{nu_2}{2}, nu_2]$ tends to zero as n tends to infinity.

Consider now the convergence of moments when all exponents m_i are even. On each portion of the coordinates $a_i < s_1^i < \dots < s_{m_i}^i < b_i$ we make the following change of variables:

$$u_{2k}^i = s_{2k}^i \quad \text{and} \quad u_{2k-1}^i = n(s_{2k}^i - s_{2k-1}^i), \quad \text{where} \quad 1 \leq k \leq m_i/2$$

with the convention $u_0^i = s_0^i = a_i$. The idea is to couple each variable with an odd subindex with the next one. In this way we obtain

$$\mathbb{E}(G_n) = \mathbf{m}! n^K \mathbb{E} \left(\int_{D_{\mathbf{m}}^n} \prod_{i=1}^N \prod_{k=1}^{\frac{m_i}{2}} f(n^H B(u_{2k}^i)) f(n^H B(u_{2k}^i - \frac{u_{2k-1}^i}{n})) du \right), \quad (3.23)$$

where K and $D_{\mathbf{m}}^n$ are as follows:

$$K = \frac{|\mathbf{m}|Hd}{2}$$

and

$$D_{\mathbf{m}}^n = \left\{ u \in \mathbb{R}^{|\mathbf{m}|} : a_i < u_2^i < u_4^i < \dots < u_{m_i}^i < b_i; \right. \\ \left. 0 < u_{2k-1}^i < n(u_{2k}^i - u_{2k-2}^i), 1 \leq k \leq \frac{m_i}{2} \right\}.$$

We compute the expectation (3.23) in the following way. Define the \mathbf{m} -dimensional Gaussian random vector $X(u)$ by

$$X_{2k}^i(u) = B(u_{2k}^i) \quad \text{and} \quad X_{2k-1}^i(u) = n^H \left(B(u_{2k}^i - \frac{u_{2k-1}^i}{n}) - B(u_{2k}^i) \right),$$

where $1 \leq k \leq \frac{m_i}{2}$. The covariance matrix and the probability density function of the Gaussian random vector $X(u)$ are denoted by $Q_n(u)$ and

$$p_n(x) = (2\pi)^{-\frac{|\mathbf{m}|d}{2}} (\det Q_n(u))^{-\frac{1}{2}} \exp(x Q_n(u)^{-1} x^T),$$

respectively. With the above notation we can write

$$\mathbb{E}(G_n) = \mathbf{m}! n^K \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}^n} \prod_{i=1}^N \prod_{k=1}^{\frac{m_i}{2}} f(n^H x_{2k}^i) f(n^H x_{2k}^i + x_{2k-1}^i) p_n(x) du dx.$$

Making the change of variables $y_j^i = n^H x_j^i$ if j is even, and $y_j^i = x_j^i$ if j is odd, we then obtain

$$\mathbb{E}(G_n) = \mathbf{m}! \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}^n} \prod_{i=1}^N \prod_{k=1}^{\frac{m_i}{2}} f(y_{2k}^i) f(y_{2k}^i + y_{2k-1}^i) p_n(y(n)) du dy, \quad (3.24)$$

where $y_j^i(n) = n^{-H} y_j^i$ if j is even and $y_j^i(n) = y_j^i$ if j is odd.

Proposition 3.4 *Suppose that all exponents m_i are even. Then*

$$\lim_{n \rightarrow \infty} \mathbb{E}(G_n) = C_{\frac{|\mathbf{m}|}{2}, d} \|f\|_{\frac{1}{H}-d}^{|\mathbf{m}|} \mathbb{E} \left(\prod_{i=1}^N (W(L_{b_i}(0)) - W(L_{a_i}(0)))^{m_i} \right). \quad (3.25)$$

Proof. The proof will be done in several steps.

Step 1 Notice that we can find a sequence of functions f_N , which are infinitely differentiable with compact support, such that $\int_{\mathbb{R}^d} f_N(x) dx = 0$ and

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}^d} |f(x) - f_N(x)| \left(|x|^{\frac{1}{H}-d} \vee 1 \right) dx = 0.$$

So, by Proposition 3.1, we can assume that f is infinitely differentiable with compact support and $\int_{\mathbb{R}^d} f(x) dx = 0$.

The equation (3.24) can be written as

$$\mathbb{E}(G_n) = \mathbf{m}! \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m}}^n} F(y) p_n(y(n)) du dy, \quad (3.26)$$

where

$$F(y) = \prod_{i=1}^N \prod_{k=1}^{\frac{m_i}{2}} f(y_{2k}^i) f(y_{2k}^i + y_{2k-1}^i).$$

Let us compute the limit of the density $p_n(y(n))$ as n tends to infinity. We split the random vector $X(u)$ into two random vectors $X(u) = (Y(u), Z_n(u))$, where $Y(u)$ contains the components of $X(u)$ with even subindices, and $Z_n(u)$ contains the components with odd subindices. That is, $Y(u)$ is an $\frac{|\mathbf{m}|d}{2}$ -dimensional random vector, such that $Y_k^i(u) = B(u_{2k}^i)$ for $1 \leq i \leq N$ and $1 \leq k \leq \frac{m_i}{2}$. We denote by $A(u)$ the covariance matrix of $Y(u)$, which does not depend on n . On the other hand, the covariance matrix between the components of $Z_n(u)$ and $Y(u)$ converges to the zero matrix, and the covariance matrix of the random vector $Z(u)$ converges to a diagonal matrix with entries equal to $(u_{2k-1}^i)^{2H}$, $1 \leq i \leq N$, and $1 \leq k \leq \frac{m_i}{2}$. Therefore,

$$\lim_{n \rightarrow \infty} p_n(y(n)) = (2\pi)^{-\frac{|\mathbf{m}|d}{2}} (\det A(u))^{-\frac{1}{2}} \prod_{i=1}^N \prod_{k=1}^{\frac{m_i}{2}} (u_{2k-1}^i)^{-Hd} \exp\left(-\frac{|y_{2k-1}^i|^2}{2u_{2k-1}^i}\right).$$

On the other hand, the region $D_{\mathbf{m}}^n$ converges, as n tends to infinity, to

$$\left\{ u \in \mathbb{R}^{|\mathbf{m}|} : a_i < u_2^i < \dots < u_{m_i}^i < b_i; 0 < u_{2k-1} < \infty; 1 \leq k \leq \frac{m_i}{2}, 1 \leq i \leq N \right\}.$$

Notice that we can add a term -1 because $\int_{\mathbb{R}^d} F(y) dy_{2k-1}^i = 0$ for any i, k , and

$$\int_0^\infty u^{-Hd} \left[e^{-\frac{|y_{2k-1}^i|^2}{2u_{2k-1}^i}} - 1 \right] du = -|y_{2k-1}^i|^{\frac{1}{H}-d} \int_0^\infty u^{-Hd} \left[1 - e^{-\frac{1}{2u_{2k-1}^i}} \right] du.$$

Therefore, provided that we can interchange the limit with the integrals in the expression (3.26), we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E}(G_n) = \mathbf{m}! 2^{-\frac{|\mathbf{m}|}{2}} (2\pi)^{-\frac{|\mathbf{m}|d}{4}} C_{H,d}^{\frac{|\mathbf{m}|}{2}} \|f\|_{\frac{1}{H}-d}^{|\mathbf{m}|} \int_{O_{\frac{\mathbf{m}}{2}}} (\det A(w))^{-\frac{1}{2}} dw, \quad (3.27)$$

where

$$O_{\frac{\mathbf{m}}{2}} = \left\{ w \in \mathbb{R}^{\frac{|\mathbf{m}|}{2}} : a_i < w_1^i < \dots < w_{\frac{m_i}{2}}^i < b_i, 1 \leq i \leq N \right\},$$

and $A(w) = A(u)$ with the change of variable $w_k^i = u_{2k}^i$. Finally, the right-hand side of (3.27) can also be written as

$$\left(\prod_{i=1}^N \frac{m_i!}{2^{\frac{m_i}{2}} \frac{m_i}{2}!} C_{H,d}^{\frac{m_i}{2}} \|f\|_{\frac{1}{H}-d}^{m_i} \right) \int_{\prod_{i=1}^N [a_i, b_i]^{\frac{m_i}{2}}} (2\pi)^{-\frac{m_i d}{4}} (\det A(w))^{-\frac{1}{2}} dw, \quad (3.28)$$

and, taking into account Lemma 2.1, this would finish the proof.

Step 2 In order to justify the passage of the limit inside the integrals, we decompose the region $D_{\mathbf{m}}^n$ into two components as follows. For $K > 0$, we define

$$D_{\mathbf{m},K,1}^n = \left\{ u \in D_{\mathbf{m}}^n : 0 < u_{2k-1}^i < K \wedge n(u_{2k}^i - u_{2k-2}^i); 1 \leq k \leq \frac{m_i}{2} \right\}$$

and $D_{\mathbf{m},K,2}^n = D_{\mathbf{m}}^n - D_{\mathbf{m},K,1}^n$. Then, $\mathbb{E}(G_n) = I_{n,K}^1 + I_{n,K}^2$, where

$$I_{n,K}^1 = \mathbf{m}! \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m},K,1}^n} F(y) p_n(y(n)) du dy,$$

$$I_{n,K}^2 = \mathbf{m}! \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m},K,2}^n} F(y) p_n(y(n)) du dy.$$

The region $D_{\mathbf{m},K,1}^n$ is uniformly bounded in n and we can then interchange the limit and the integral with respect to u , provided that we have a uniform integrability condition. To do this we need the following estimate of the density $p_n(y(n))$.

For any $\xi \in \mathbb{R}^{|\mathbf{m}|}$ with components (ξ_j^i) , $1 \leq i \leq N$, $1 \leq j \leq m_i$, we can write

$$\begin{aligned} \langle \xi, X \rangle &= \sum_{i=1}^N \left(\sum_{j=1}^{\frac{m_i}{2}} \xi_{2j}^i \cdot B(u_{2j}^i) + \sum_{k=1}^{\frac{m_i}{2}} \xi_{2k-1}^i \cdot n^H \left(B(u_{2k}^i - \frac{u_{2k-1}^i}{n}) - B(u_{2k-2}^i) \right) \right) \\ &= \sum_{i=1}^N \sum_{k=1}^{\frac{m_i}{2}} \left(\sum_{(\ell, 2j) \geq (i, 2k)} \xi_{2j}^\ell \right) \cdot \left(B(u_{2k}^i - \frac{u_{2k-1}^i}{n}) - B(u_{2k-2}^i) \right) \\ &\quad + \sum_{i=1}^N \sum_{k=1}^{\frac{m_i}{2}} \left(\sum_{(\ell, 2j) \geq (i, 2k)} \xi_{2j}^\ell - n^H \xi_{2k-1}^i \right) \cdot \left(B(u_{2k}^i) - B(u_{2k}^i - \frac{u_{2k-1}^i}{n}) \right). \end{aligned}$$

Here we have used the ordering $(\ell, 2j) \geq (i, 2k)$ if $\ell > i$ or $\ell = i$ and $j \geq k$.

By the local nondeterminism property (2.3),

$$\begin{aligned} \text{Var} \langle \xi, X \rangle &\geq k_H \left[\sum_{i=1}^N \sum_{k=1}^{\frac{m_i}{2}} \left| \sum_{(\ell, 2j) \geq (i, 2k)} \xi_{2j}^\ell \right|^2 \cdot \left(u_{2k}^i - \frac{u_{2k-1}^i}{n} - u_{2k-2}^i \right)^{2H} \right. \\ &\quad \left. + \sum_{i=1}^N \sum_{k=1}^{\frac{m_i}{2}} \left| \sum_{(\ell, 2j) \geq (i, 2k)} \xi_{2j}^\ell - n^H \xi_{2k-1}^i \right|^2 \left(\frac{u_{2k-1}^i}{n} \right)^{2H} \right] \\ &= k_H \left[\sum_{i=1}^N \sum_{k=1}^{\frac{m_i}{2}} |\eta_{2k}^i|^2 \left(u_{2k}^i - \frac{u_{2k-1}^i}{n} - u_{2k-2}^i \right)^{2H} \right. \\ &\quad \left. + \sum_{i=1}^N \sum_{k=1}^{\frac{m_i}{2}} |\eta_{2k}^i - n^H \eta_{2k-1}^i|^2 \left(\frac{u_{2k-1}^i}{n} \right)^{2H} \right] \\ &=: k_H R(\eta), \end{aligned} \tag{3.29}$$

where we have made the change of variables

$$\eta_{2k}^i = \sum_{(\ell, 2j) \geq (i, 2k)} \xi_{2j}^\ell \quad \text{and} \quad \eta_{2k-1}^i = \xi_{2k-1}^i.$$

This implies that

$$\begin{aligned} (\det Q_n)^{-\frac{1}{2}} &= (2\pi)^{-\frac{|\mathbf{m}|d}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \exp\left(-\frac{1}{2} \text{Var} \langle \xi, X \rangle\right) d\xi \\ &\leq (2\pi)^{-\frac{|\mathbf{m}|d}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \exp\left(-\frac{k_H}{2} R(\eta)\right) d\eta \\ &= c_1 \prod_{i=1}^N \prod_{k=1}^{\frac{m_i}{2}} (u_{2k-1}^i)^{-Hd} \left(u_{2k}^i - \frac{u_{2k-1}^i}{n} - u_{2k-2}^i\right)^{-Hd}. \end{aligned}$$

Therefore,

$$p_n(y(n)) \leq c_2 \prod_{i=1}^N \prod_{k=1}^{\frac{m_i}{2}} (u_{2k-1}^i)^{-Hd} \left(u_{2k}^i - \frac{u_{2k-1}^i}{n} - u_{2k-2}^i\right)^{-Hd}. \quad (3.30)$$

As a consequence of (3.30) and the inequality (4.5) in Lemma 4.5,

$$\begin{aligned} &\int_{D_{\mathbf{m}, K, 1}^n} p_n(y(n)) du \\ &\leq c_3 \int_{D_{\mathbf{m}, K, 1}^n} \prod_{i=1}^N \prod_{k=1}^{\frac{m_i}{2}} (u_{2k-1}^i)^{-Hd} \left(u_{2k}^i - \frac{u_{2k-1}^i}{n} - u_{2k-2}^i\right)^{-Hd} du \\ &\leq c_4, \end{aligned}$$

where c_4 is a constant independent of n and y . Thus, taking into account that the function $F(y)$ is integrable, by the dominated convergence theorem we obtain

$$\lim_{n \rightarrow \infty} I_{n, K}^1 = \mathbf{m}! \int_{\mathbb{R}^{|\mathbf{m}|d}} F(y) \left(\lim_{n \rightarrow \infty} \int_{D_{\mathbf{m}, K, 1}^n} p_n(y(n)) du \right) dy.$$

On the other hand, again by (3.30) and Lemma 4.5, there exists $p > 1$ such that

$$\sup_n \int_{D_{\mathbf{m}, K, 1}^n} |p_n(y(n))|^p du < \infty, \quad (3.31)$$

which implies

$$\lim_{n \rightarrow \infty} I_{n, K}^1 = \mathbf{m}! \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{\mathbb{R}^{|\mathbf{m}|}} F(y) \lim_{n \rightarrow \infty} \mathbf{1}_{D_{\mathbf{m}, K, 1}^n}(u) p_n(y(n)) du dy.$$

With the same notation as above we get

$$\begin{aligned} \lim_{n \rightarrow \infty} I_{n, K}^1 &= \mathbf{m}! (2\pi)^{-\frac{|\mathbf{m}|d}{2}} \left(\int_{O_{\frac{\mathbf{m}}{2}}} (\det A(w))^{-\frac{1}{2}} dw \right) \\ &\times \int_{\mathbb{R}^{|\mathbf{m}|d}} \prod_{i=1}^N \prod_{k=1}^{\frac{m_i}{2}} \left(f(y_{2k}^i) f(y_{2k}^i + y_{2k-1}^i) \int_0^K u^{-Hd} \left(e^{-\frac{|y_{2k-1}^i|^2}{2u^{2H}}} - 1 \right) du \right) dy. \end{aligned} \quad (3.32)$$

The right-hand side of the above equality converges to the term in (3.28) as K tends to infinity.

Step 3 Now it suffices to show that

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} I_{n,K}^2 = 0. \quad (3.33)$$

First we observe that

$$D_{\mathbf{m},K,2}^n \subset \cup_{i=1}^N \cup_{j=1}^{\frac{m_i}{2}} D_{\mathbf{m},K,i,k}^n,$$

where

$$D_{\mathbf{m},K,i,k}^n = \{u \in D_{\mathbf{m}}^n : u_{2k-1}^i > K\}.$$

So we only need to show that for all (i, k)

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m},K,i,k}^n} F(y) p_n(y(n)) du dy = 0. \quad (3.34)$$

As a consequence of Proposition 3.3, we can replace $D_{\mathbf{m},K,i,k}^n$ in (3.34) with

$$D_{\mathbf{m},K,i,k}^{n,1} = \left\{u \in D_{\mathbf{m}}^n ; K < u_{2k-1}^i < \frac{n(u_{2k}^i - u_{2k-2}^i)}{2}\right\}$$

and just show that for all (i, k)

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m},K,i,k}^{n,1}} F(y) p_n(y(n)) du dy = 0. \quad (3.35)$$

To do this we need more refined estimates of the density $p_n(y(n))$. By Fourier analysis

$$p_n(y(n)) = (2\pi)^{-|\mathbf{m}|d} \int_{\mathbb{R}^{|\mathbf{m}|d}} \exp\left(-\frac{1}{2} \text{Var} \langle \xi, X \rangle - \iota \sum_{i=1}^N \sum_{j=1}^{\frac{m_i}{2}} \left(\frac{\xi_{2j}^i \cdot y_{2j}^i}{n^H} + \xi_{2j-1}^i \cdot y_{2j-1}^i\right)\right) d\xi.$$

We choose a set J of indexes of the form $(i, 2j-1)$, where $1 \leq i \leq N$ and $1 \leq j \leq \frac{m_i}{2}$. For each index in J we introduce the operator

$$\Delta_{i,2j-1} F(y_{2j-1}^i) = F(y_{2j-1}^i) - F(0)$$

and set $\Delta_J = \prod_{(i,2j-1) \in J} \Delta_{i,2j-1}$. Taking into account that the integral on the variable y_{2j-1}^i is zero, we can replace $p_n(y(n))$ in (3.35) by $\Delta_J p_n(y(n))$. Using (3.29), we obtain the following estimate

$$\begin{aligned} \Delta_J p_n(y(n)) &\leq c_5 \int_{\mathbb{R}^{|\mathbf{m}|d}} \exp\left(-\frac{k_H}{2} \sum_{i=1}^N \sum_{j=1}^{\frac{m_i}{2}} \left[|\eta_{2j}^i|^2 \left(u_{2j}^i - \frac{u_{2j-1}^i}{n} - u_{2j-2}^i\right)^{2H} \right. \right. \\ &\quad \left. \left. + |\eta_{2j}^i - n^H \eta_{2j-1}^i|^2 \left(\frac{u_{2j-1}^i}{n}\right)^{2H}\right]\right) \prod_{i=1}^N \prod_{j=1}^{\frac{m_i}{2}} |e^{-\iota \eta_{2j-1}^i \cdot y_{2j-1}^i} - 1| d\eta \\ &= c_5 n^{-\frac{|\mathbf{m}|Hd}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \exp\left(-\frac{k_H}{2} \sum_{i=1}^N \sum_{j=1}^{\frac{m_i}{2}} \left[|\eta_{2j}^i|^2 \left(u_{2j}^i - \frac{u_{2j-1}^i}{n} - u_{2j-2}^i\right)^{2H} \right. \right. \\ &\quad \left. \left. + |\eta_{2j-1}^i|^2 \left(\frac{u_{2j-1}^i}{n}\right)^{2H}\right]\right) \prod_{i=1}^N \prod_{j=1}^{\frac{m_i}{2}} |e^{-\iota \frac{(\eta_{2j}^i - \eta_{2j-1}^i) \cdot y_{2j-1}^i}{n^H}} - 1| d\eta. \end{aligned} \quad (3.36)$$

This shows that

$$\begin{aligned}
& \left| \int_{\mathbb{R}^{|\mathbf{m}|d}} \int_{D_{\mathbf{m},K,i,k}^{n,1}} F(y) p_n(y(n)) du dy \right| \\
& \leq c_6 n^{-\frac{|\mathbf{m}|Hd}{2}} \int_{\mathbb{R}^{|\mathbf{m}|d}} |F(y)| \int_{D_{\mathbf{m},K,i,k}^{n,1}} \int_{\mathbb{R}^{|\mathbf{m}|d}} \exp \left(-\frac{k_H}{2} \sum_{i=1}^N \sum_{j=1}^{\frac{m_i}{2}} \left[|\eta_{2j}^i|^2 \left(u_{2j}^i - \frac{u_{2j-1}^i}{n} - u_{2j-2}^i \right)^{2H} \right. \right. \\
& \quad \left. \left. + |\eta_{2j-1}^i|^2 \left(\frac{u_{2j-1}^i}{n} \right)^{2H} \right] \right) \prod_{i=1}^N \prod_{j=1}^{\frac{m_i}{2}} \left| e^{-\iota \frac{(\eta_{2j}^i - \eta_{2j-1}^i) \cdot y_{2j-1}^i}{n^H}} - 1 \right| d\eta du dy. \tag{3.37}
\end{aligned}$$

To estimate the right-hand side of (3.37), we first consider the integral in the variables $u = u_{2k}^i$, $v = u_{2k-1}^i$, $w = \eta_{2k}^i$ and $z = \eta_{2k-1}^i$. Set $u_{2k-2}^i = u_0$ and $y_{2k-1}^i = y$. That is, we have the integral

$$\begin{aligned}
I_1(u_0, b_i, y) &= n^{-Hd} \int_{u_0}^{b_i} \int_K \int_{\mathbb{R}^{2d}} \exp \left(-\frac{\kappa_H}{2} \left(|z|^2 \left(\frac{v}{n} \right)^{2H} + |w|^2 \left(u - u_0 - \frac{v}{n} \right)^{2H} \right) \right) \\
&\quad \times \left(\left| e^{-\iota \frac{z \cdot y}{n^H}} - 1 \right| + \left| e^{-\iota \frac{w \cdot y}{n^H}} - 1 \right| \right) dw dz dv du.
\end{aligned}$$

We see that

$$I_1(u_0, b_i, y) = I_2(u_0, b_i, y) + I_3(u_0, b_i, y),$$

where

$$\begin{aligned}
I_2(u_0, b_i, y) &= n^{-Hd} \int_{u_0}^{b_i} \int_K \int_{\mathbb{R}^{2d}} \exp \left(-\frac{\kappa_H}{2} \left(|z|^2 \left(\frac{v}{n} \right)^{2H} + |w|^2 \left(u - u_0 - \frac{v}{n} \right)^{2H} \right) \right) \\
&\quad \times \left| e^{-\iota \frac{z \cdot y}{n^H}} - 1 \right| dw dz dv du
\end{aligned}$$

and

$$\begin{aligned}
I_3(u_0, b_i, y) &= n^{-Hd} \int_{u_0}^{b_i} \int_K \int_{\mathbb{R}^{2d}} \exp \left(-\frac{\kappa_H}{2} \left(|z|^2 \left(\frac{v}{n} \right)^{2H} + |w|^2 \left(u - u_0 - \frac{v}{n} \right)^{2H} \right) \right) \\
&\quad \times \left| e^{-\iota \frac{w \cdot y}{n^H}} - 1 \right| dw dz dv du.
\end{aligned}$$

Integrating with respect to w and using the inequality (4.4) in the appendix lead us to

$$\begin{aligned}
I_2(u_0, b_i, y) &= c n^{-Hd} \int_{u_0}^{b_i} \int_K \int_{\mathbb{R}^d} \left(u - u_0 - \frac{v}{n} \right)^{-Hd} e^{-\frac{\kappa_H |z|^2}{2} \left(\frac{v}{n} \right)^{2H}} \left| e^{-\iota \frac{z \cdot y}{n^H}} - 1 \right| dz dv du \\
&\leq c K^{1-Hd-H\beta} |y|^\beta \int_{u_0}^{b_i} (u - u_0)^{-Hd} du \\
&\leq c K^{1-Hd-H\beta} (b_i - u_0)^{1-Hd} |y|^\beta. \tag{3.38}
\end{aligned}$$

In a similar way but with the application of (4.3) instead of (4.4) we obtain

$$\begin{aligned}
I_3(u_0, b_i, y) &= c \int_{u_0}^{b_i} \int_K \int_{\mathbb{R}^d} v^{-H} e^{-\frac{\kappa_H |w|^2}{2} \left(u - u_0 - \frac{v}{n} \right)^{2H}} \left| e^{-\iota \frac{w \cdot y}{n^H}} - 1 \right| dw dv du \\
&\leq c n^{1-Hd-H\beta} |y|^\beta \int_{u_0}^{b_i} (u - u_0)^{1-2Hd-H\beta} du \\
&\leq c n^{1-Hd-H\beta} (b_i - u_0)^{2-2Hd-H\beta} |y|^\beta. \tag{3.39}
\end{aligned}$$

Combining (3.38) and (3.39) gives

$$I_1(u_0, b_i, y) \leq c K^{1-Hd-H\beta} (b_i - a_i)^{1-Hd} |y|^\beta + c n^{1-Hd-H\beta} (b_i - a_i)^{2-2Hd-H\beta} |y|^\beta. \quad (3.40)$$

Once this is done, we proceed to consider the integrals in the variables $u = u_{2l}^j$, $v = u_{2l-1}^j$, $z = \eta_{2l}^j$ and $w = \eta_{2l-1}^j$ with indices $(j, l) \neq (i, k)$. Set also $u_{2l-2}^j = u_0$ and $y_{2l-1}^j = y$. That is, we have the integral

$$\begin{aligned} I_4(u_0, b_j, y) &= n^{-Hd} \int_{u_0}^{b_j} \int_0^{n(u-u_0)} \int_{\mathbb{R}^{2d}} \exp \left(-\frac{\kappa_H}{2} \left(|z|^2 \left(\frac{v}{n} \right)^{2H} + |w|^2 \left(u - u_0 - \frac{v}{n} \right)^{2H} \right) \right) \\ &\quad \times \left(\left| e^{-\iota \frac{z \cdot y}{n^H}} - 1 \right| + \left| e^{-\iota \frac{w \cdot y}{n^H}} - 1 \right| \right) dw dz dv du. \end{aligned}$$

We can decompose this integral into two components:

$$I_4(u_0, b_j, y) = I_5(u_0, b_j, y) + I_6(u_0, b_j, y),$$

where

$$\begin{aligned} I_5(u_0, b_j, y) &= n^{-Hd} \int_{u_0}^{b_j} \int_0^{n(u-u_0)} \int_{\mathbb{R}^{2d}} \exp \left(-\frac{\kappa_H}{2} \left(|z|^2 \left(\frac{v}{n} \right)^{2H} + |w|^2 \left(u - u_0 - \frac{v}{n} \right)^{2H} \right) \right) \\ &\quad \times \left| e^{-\iota \frac{z \cdot y}{n^H}} - 1 \right| dw dz dv du \\ &= c n^{-Hd} \int_{u_0}^{b_j} \int_0^{n(u-u_0)} \left(u - u_0 - \frac{v}{n} \right)^{-Hd} \int_{\mathbb{R}} e^{-\frac{\kappa_H |z|^2}{2} \left(\frac{v}{n} \right)^{2H}} \left| e^{-\iota \frac{z \cdot y}{n^H}} - 1 \right| dz dv du \end{aligned}$$

and

$$\begin{aligned} I_6(u_0, b_j, y) &= n^{-Hd} \int_{u_0}^{b_j} \int_0^{n(u-u_0)} \int_{\mathbb{R}^{2d}} \exp \left(-\frac{\kappa_H}{2} \left(|z|^2 \left(\frac{v}{n} \right)^{2H} + |w|^2 \left(u - u_0 - \frac{v}{n} \right)^{2H} \right) \right) \\ &\quad \times \left| e^{-\iota \frac{w \cdot y}{n^H}} - 1 \right| dw dz dv du \\ &= c \int_{u_0}^{b_j} \int_0^{n(u-u_0)} v^{-Hd} \int_{\mathbb{R}^d} e^{-\frac{\kappa_H |w|^2}{2} \left(u - u_0 - \frac{v}{n} \right)^{2H}} \left| e^{-\iota \frac{w \cdot y}{n^H}} - 1 \right| dw dv du. \end{aligned}$$

The inequalities (4.2) and (4.1) imply that

$$\begin{aligned} I_4(u_0, b_j, y) &\leq c \int_{u_0}^{b_j} (u - u_0)^{-Hd} |y|^{\frac{1}{H}-d} du \\ &= c (b_j - u_0)^{1-Hd} |y|^{\frac{1}{H}-d} \\ &\leq c (b_j - a_j)^{1-Hd} |y|^{\frac{1}{H}-d}. \end{aligned}$$

The remaining integrals can be dealt with in the same way as $I_4(u_0, b_j, y)$. Thus the statement (3.35) follows. The proof is completed. \blacksquare

Proof of Theorem 1.1. This follows from Lemma 2.1, Propositions 3.1, 3.2 and 3.4 by the method of moments.

Remark Although the constant $C_{H,d}$ is finite for $H > \frac{1}{d+2}$, the proof of Theorem 1.1 only works for $H > \frac{1}{d+1}$. The reason for this is that for any $y \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \exp\left(-\frac{\kappa}{2}|\xi|^2 u^{2H}\right) (1 - e^{\iota\xi \cdot y}) d\xi = \left(\frac{2\pi}{\kappa}\right)^{\frac{d}{2}} u^{-Hd} \left(1 - \exp\left(-\frac{|y|^2}{2\kappa u^{2H}}\right)\right),$$

which is bounded by $c|y|^2 u^{-H(d+2)}$, while, on the other hand,

$$\int_{\mathbb{R}^d} \exp\left(-\frac{\kappa}{2}|\xi|^2 u^{2H}\right) |e^{\iota\xi \cdot y} - 1| d\xi \leq c|y| u^{-H(d+1)}.$$

So, any type of estimation procedure, like the one based on the local nondeterminism property used in this paper, will lead to an upper bound of the form $u^{-H(d+1)}$.

4 Appendix

Here we give some lemmas which are necessary in the proof of Theorem 1.1.

Lemma 4.1 *Let $0 < \beta < 1$. If $f \in H_0^\beta$, then $\|f\|_\beta^2$ given in (1.2) is well-defined and*

$$\|f\|_\beta^2 = c_\beta^{-1} \int_{\mathbb{R}^d} |\mathcal{F}f(\xi)|^2 |\xi|^{-\beta-d} d\xi \geq 0,$$

where $\mathcal{F}f(\xi)$ denotes the Fourier transform of f and

$$c_{\beta,d} = \int_{\mathbb{R}^d} (1 - \cos(x \cdot \xi)) |\xi|^{-\beta-d} d\xi > 0$$

is independent of x if $|x| = 1$.

Proof. For any $x \in S^{d-1}$ and any $d \times d$ orthogonal matrix Q , the change of variable $\xi = Q\eta$ yields

$$\int_{\mathbb{R}^d} (1 - \cos(x \cdot \xi)) |\xi|^{-\beta-d} d\xi = \int_{\mathbb{R}^d} (1 - \cos((Q^T x) \cdot \eta)) |\eta|^{-\beta-d} d\eta > 0.$$

This shows that $\int_{\mathbb{R}^d} (1 - \cos(x \cdot \xi)) |\xi|^{-\beta-d} d\xi$ depends only on $|x|$. The substitution $\xi = \frac{1}{|x|}\eta$ gives us $\int_{\mathbb{R}^d} (1 - \cos(x \cdot \xi)) |\xi|^{-\beta-d} d\xi = c_{\beta,d} |x|^\beta$. Then an elementary result from Fourier analysis [12] yields

$$\begin{aligned} \|f\|_\beta^2 &= c_\beta^{-1} \int_{\mathbb{R}^{2d}} f(x) f(y) \left(\int_{\mathbb{R}^d} (e^{\iota(x-y) \cdot \xi} - 1) |\xi|^{-\beta-d} d\xi \right) dx dy \\ &= c_\beta^{-1} \int_{\mathbb{R}^d} |\mathcal{F}f(\xi)|^2 |\xi|^{-\beta-d} d\xi \geq 0. \end{aligned}$$

■

Lemma 4.2 *Assume that $Hd < 1$. Let X be a d -dimensional centered normal random vector with variance σ^2 . Then, for any $n \in \mathbb{N}$ and $y \in \mathbb{R}^d$, there exists a constant c depending only on H , d and σ such that*

$$\int_0^\infty u^{-Hd} \mathbb{E} \left| \exp\left(\iota \frac{y \cdot X}{n^H u^H}\right) - 1 \right| du \leq c n^{Hd-1} |y|^{\frac{1}{H}-d}.$$

Proof. It suffices to show the above inequality when $y \neq 0$. Making the change of variable $v = |y|^{-\frac{1}{H}} nu$ gives

$$\begin{aligned}
& \int_0^\infty u^{-Hd} \mathbb{E} \left| \exp \left(\iota \frac{y \cdot X}{n^H u^H} \right) - 1 \right| du \\
&= n^{Hd-1} |y|^{\frac{1}{H}-d} \int_0^\infty v^{-Hd} \mathbb{E} \left| \exp \left(\iota \frac{y \cdot X}{|y| v^H} \right) - 1 \right| dv \\
&\leq n^{Hd-1} |y|^{\frac{1}{H}-d} \int_0^\infty v^{-Hd} \left(2 \wedge v^{-H} \mathbb{E} |X| \right) dv \\
&= c n^{Hd-1} |y|^{\frac{1}{H}-d}.
\end{aligned}$$

■

Lemma 4.3 Assume $1 - H \leq Hd < 1$ and $0 \leq \beta \leq 1$ with $H\beta < 1 - Hd$. Then, for all $n \in \mathbb{N}$, there exists a constant c independent of n such that

$$\int_0^{ns} u^{-Hd-H\beta} du \int_{\mathbb{R}^d} e^{-\kappa|\xi|^2(s-\frac{u}{n})^{2H}} |1 - e^{\iota \frac{\xi \cdot y}{n^H}}| d\xi \leq c n^{-H\beta} s^{-Hd-H\beta} |y|^{\frac{1}{H}-d}, \quad (4.1)$$

$$n^{-Hd-H\beta} \int_0^{ns} \left(s - \frac{u}{n}\right)^{-Hd-H\beta} du \int_{\mathbb{R}^d} e^{-\kappa|\xi|^2(\frac{u}{n})^{2H}} |1 - e^{\iota \frac{\xi \cdot y}{n^H}}| d\xi \leq c n^{-H\beta} s^{-Hd-H\beta} |y|^{\frac{1}{H}-d}, \quad (4.2)$$

where κ is a positive constant.

Proof. We first note that (4.1) follows easily from (4.2) by the change of variable. So, it suffices to prove (4.2). Denote the left hand side of (4.2) by I . We then have

$$\begin{aligned}
I &= n^{-Hd-H\beta} \int_0^{ns} \left(s - \frac{u}{n}\right)^{-Hd-H\beta} du \int_{\mathbb{R}^d} e^{-\kappa|\xi|^2(\frac{u}{n})^{2H}} |1 - e^{\iota \frac{\xi \cdot y}{n^H}}| d\xi \\
&= n^{-H\beta} \int_0^{ns} \left(s - \frac{u}{n}\right)^{-Hd-H\beta} u^{-Hd} \int_{\mathbb{R}^d} e^{-\kappa|x|^2} |1 - e^{\iota \frac{x \cdot y}{u^H}}| dx du,
\end{aligned}$$

where we made the change of variable $\xi(\frac{u}{n})^H = x$.

By Lemma 4.2, we obtain

$$\begin{aligned}
& \int_0^{\frac{ns}{2}} \left(s - \frac{u}{n}\right)^{-Hd-H\beta} u^{-Hd} \int_{\mathbb{R}^d} e^{-\kappa|x|^2} |1 - e^{\iota \frac{x \cdot y}{u^H}}| dx du \\
&\leq c_1 s^{-Hd-H\beta} \int_0^\infty u^{-Hd} \int_{\mathbb{R}^d} e^{-\kappa|x|^2} |1 - e^{\iota \frac{x \cdot y}{u^H}}| dx du \\
&= c_2 s^{-Hd-H\beta} |y|^{\frac{1}{H}-d}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \int_{\frac{ns}{2}}^{ns} \left(s - \frac{u}{n}\right)^{-Hd-H\beta} u^{-Hd} \int_{\mathbb{R}^d} e^{-\kappa|x|^2} |1 - e^{t\frac{x \cdot y}{u^H}}| dx du \\
& \leq \int_{\frac{ns}{2}}^{ns} \left(s - \frac{u}{n}\right)^{-Hd-H\beta} u^{-Hd-H\beta_1} \int_{\mathbb{R}^d} e^{-\kappa|x|^2} |x \cdot y|^{\beta_1} dx du \\
& \leq c_3 (ns)^{-Hd-H\beta_1} |y|^{\beta_1} \int_{\frac{ns}{2}}^{ns} \left(s - \frac{u}{n}\right)^{-Hd-H\beta} du \\
& \leq c_4 n^{1-Hd-H\beta_1} s^{1-H\beta-2Hd-H\beta_1} |y|^{\beta_1},
\end{aligned}$$

where β_1 can be any constant in $[0, 1]$ and we used $H\beta < 1 - Hd$ in the last inequality. Our result follows by choosing $\beta_1 = \frac{1}{H} - d$. This is possible because $1 - H \leq Hd < 1$. \blacksquare

Lemma 4.4 *For $n \in \mathbb{N}$, we assume that $1 - H < Hd < 1$ and $0 < K < \frac{ns}{2}$. Then there exists a constant c independent of n and K such that*

$$\int_K^{\frac{ns}{2}} u^{-Hd} du \int_{\mathbb{R}^d} e^{-\kappa|\xi|^2(s-\frac{u}{n})^{2H}} |1 - e^{t\frac{\xi \cdot y}{n^H}}| d\xi \leq c n^{1-Hd-H\beta} s^{1-2Hd-H\beta} |y|^\beta, \quad (4.3)$$

$$n^{-Hd} \int_K^{\frac{ns}{2}} \left(s - \frac{u}{n}\right)^{-Hd} du \int_{\mathbb{R}^d} e^{-\kappa|\xi|^2(\frac{u}{n})^{2H}} |1 - e^{t\frac{\xi \cdot y}{n^H}}| d\xi \leq c s^{-Hd} K^{1-Hd-H\beta} |y|^\beta, \quad (4.4)$$

where κ and β are positive constants with $1 - Hd < H\beta < H \wedge (2 - 2Hd)$.

Proof. The inequality (4.3) follows easily from the proof of Lemma 4.3. We shall show (4.4). Denote the left hand side of (4.4) by I . Then we have

$$\begin{aligned}
I &= n^{1-Hd} \int_{K/n}^{s/2} (s - u)^{-Hd} du \int_{\mathbb{R}^d} e^{-\kappa|\xi|^2 u^{2H}} |1 - e^{t\frac{\xi \cdot y}{n^H}}| d\xi \\
&\leq c |y|^\beta n^{1-Hd-H\beta} \int_{K/n}^{s/2} (s - u)^{-Hd} u^{-Hd-H\beta} du.
\end{aligned}$$

Since $\frac{K}{n} < \frac{s}{2}$ and $1 - Hd < H\beta$, we have

$$\int_{K/n}^{s/2} (s - u)^{-Hd} u^{-Hd-H\beta} du \leq c s^{-Hd} \int_{K/n}^{s/2} u^{-Hd-H\beta} du \leq c s^{-Hd} \left(\frac{K}{n}\right)^{1-Hd-H\beta}.$$

Therefore,

$$I \leq c s^{-Hd} K^{1-Hd-H\beta} |y|^\beta.$$

This proves the lemma. \blacksquare

Lemma 4.5 *For any $K > 0$ and $n \in \mathbb{N}$, there exist constants c_1 and c_2 independent of n such that*

$$\int_0^{K \wedge nu} v^{-Hd} \left(u - \frac{v}{n}\right)^{-Hd} dv \leq c_1 K^{1-Hd} u^{-Hd}, \quad (4.5)$$

and

$$\int_0^u v^{-Hd} (u - v)^{-Hd} dv = c_2 u^{1-2Hd}. \quad (4.6)$$

Proof. The inequality (4.6) follows easily from

$$\int_0^u v^{-Hd} (u-v)^{-Hd} dv = u^{1-2Hd} \int_0^1 w^{-Hd} (1-w)^{-Hd} dw.$$

We only need to show (4.5). Notice that

$$\int_0^{K \wedge nu} v^{-Hd} \left(u - \frac{v}{n}\right)^{-Hd} dv = u^{-Hd} (nu)^{1-Hd} \int_0^{\frac{K}{nu} \wedge 1} v^{-Hd} (1-v)^{-Hd} dv.$$

If $nu \leq 2K$, then

$$\int_0^{\frac{K}{nu} \wedge 1} v^{-Hd} (1-v)^{-Hd} dv \leq \int_0^1 v^{-Hd} (1-v)^{-Hd} dv.$$

If $nu > 2K$, then

$$\int_0^{\frac{K}{nu} \wedge 1} v^{-Hd} (1-v)^{-Hd} dv \leq 2 \int_0^{\frac{K}{nu}} v^{-Hd} dv \leq \frac{2}{1-Hd} \left(\frac{K}{nu}\right)^{1-Hd}.$$

Therefore,

$$\int_0^{K \wedge nu} v^{-Hd} \left(u - \frac{v}{n}\right)^{-Hd} dv \leq c_3 K^{1-Hd} u^{-Hd}.$$

■

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